EXPONENTIAL SMOOTHING

Exponential smoothing is the most widely used class of procedures for smoothing discrete time series in order to forecast the immediate future. The idea of exponential smoothing is to smooth the original series the way the moving average does and to use the smoothed series in forecasting future values of the variable of interest. In exponential smoothing, however, we want to allow the more recent values of the series to have greater influence on the forecast of future values than the more distant observations.

Exponential smoothing is a simple and pragmatic approach to forecasting, whereby the forecast is constructed from an **exponentially weighted average** of past observations. The largest weight is given to the present observation, less weight to the immediately preceding observation, even less weight to the observation before that, and so on (exponential decay of influence of past data)

SIMPLE EXPONENTIAL SMOOTHING

This forecasting method is most widely used of all forecasting techniques. It requires little computation. This method is used when data pattern is approximately horizontal (i.e., there is no neither cyclic variation nor pronounced trend in the historical data).

Let an observed time series be y1, y2, …. yn. Formally, the simple exponential smoothing equation takes the form of

$$S_{t+1} = \alpha y_t + (1-\alpha)\, S_t$$

Where S_i → The smoothed value of time series at time i

y_i → Actual value of time series at time i

α →Smoothing constant

In case of simple exponential smoothing, the smoothed statistic is the Forecasted value.

$$F_{t+1} = \alpha y_t + (1-\alpha) F_t$$

Where $F_{t+1} \rightarrow$ Forecasted value of time series at time t+1

Ft $\rightarrow$ Forecasted value of time series at time t

This means:

$F_t = \alpha y_{t-1} + (1-\alpha) F_{t-1}$

$F_{t-1} = \alpha y_{t-1} + (1-\alpha) F_{t-2}$

$F_{t-2} = \alpha y_{t-2} + (1-\alpha) F_{t-3}$

$F_{t-3} = \alpha y_{t-3} + (1-\alpha) F_{t-4}$

Substituting, $F_{t+1} = \alpha y_t + (1-\alpha) F_t = \alpha y_t + (1-\alpha)(\alpha y_{t-1} + (1-\alpha)F_{t-1}) =$

$= \alpha y_t + \alpha (1-\alpha) y_{t-1} + (1-\alpha)^2 F_{t-1} =$

$= \alpha y_t + \alpha (1-\alpha) y_{t-1} + \alpha (1-\alpha)^2 y_{t-2} + (1-\alpha)^3 F_{t-2}$

$= \alpha y_t + \alpha (1-\alpha) y_{t-1} + \alpha (1-\alpha)^2 y_{t-2} + \alpha(1-\alpha)^3 y_{t-3} + (1-\alpha)^4 F_{t-3}$

Generalizing,

$$F_{t+1} = \sum_{i=0}^{t-1} \alpha(1-\alpha)^i y_{t-i} + (1-\alpha)^t F_1$$

The series of weights used in producing the forecast F_t are α , $\alpha (1-\alpha)$, $\alpha(1-\alpha)^2$, $\alpha(1-\alpha)^3$…. These weights decline toward zero in an exponential fashion; thus, as we go back in the series, each value has a smaller weight in terms of its effect on the forecast. The exponential decline of the weights towards zero is evident.

<u>Choosing α:</u>

After the model is specified, its performance characteristics should be verified or validated by comparison of its forecast with historical data for the process it was designed to forecast.

We can use the error measures such as *MAPE* (Mean absolute percentage error), *MSE* (Mean square error) or *RMSE* (Root mean square error) and α is chosen such that the error is minimum.

Usually the *MSE* or *RMSE* can be used as the criterion for selecting an appropriate smoothing constant. For instance, by assigning a values from 0.1 to 0.99, we select the value that produces the smallest *MSE* or *RMSE*

Since F1 is not known, we can:

- Set the first estimate equal to the first observation. Thus we can use 1
- Use the average of the first five or six observations for the initial smoothed value.

DOUBLE EXPONENTIAL SMOOTHIG-HOLT'S TREND METHOD

Under the assumption of no trend in the data, simple exponential smoothing yields goods results but it fails in case of existence of trend. Double exponential smoothing is used when there is a linear trend in the data.

The basic idea behind double exponential smoothing is to introduce a term to take into account the possibility of a series exhibiting some form of trend. This slope component is itself updated via exponential smoothing.

Suppose the data exhibits a linear trend as:

$y_t = b_0 + b_1 t + e_t$

where, b_0 and b_1 may change slowly with time.

The basic equations for Holt's Method are:

1. $\mu_t = \alpha y_t + (1 - \alpha)(\mu_{t-1} + T_{t-1})$
2. $T_t = \beta(\mu_t - \mu_{t-1}) + (1 - \beta)T_{t-1}$
3. $F_{t+m} = \mu_t + mT_t$

where

μ_t → *Exponentially smoothed value of the series at time t*

y_t →Actual observation of time series at time t

Tt →Trend Estimate

α→Exponential Smoothing Constant for the data

β→Smoothing constant for trend

F_{t+m}→m period ahead forecasted value

The difference between 2 successive exponential smoothing values is $(\mu_t - \mu_{t-1})$ used as an estimate of the trend. The estimate of the trend is smoothed by multiplying it by β and then multiplying the old estimate of the trend by $(1 - \beta)$. To forecast, the trend is multiplied by the number of periods ahead that one desires to forecast and then the product is added to μ_t.

<u>Choice of α and β</u>

Choose one that minimize MSE or MAPE.

<u>Initialization</u>

Level : $\mu_1 = y_1$

Trend : $T_1 =(y_2 – y_1)/(T_2-T_1)$ or $(y_4 – y_1)/(T_4-T_1)$

TRIPLE EXPONENTIAL SMOOTHING HOLT'S WINTERS TREND AND SEASONALITY METHOD:

Under the assumption of presence of only linear trend in the data, double exponential smoothing yields goods results but it fails in case of existence of trend and seasonality. Triple exponential smoothing is used when there is trend in the data along with seasonal variations.

Two Holt-Winters methods are designed for time series that exhibit linear trend

1. **Additive Holt-Winters method**: used for time series with constant (additive) seasonal variations
2. **Multiplicative Holt-Winters method**: used for time series with increasing (multiplicative) seasonal variations

<u>Holt- Winter's Trend and Seasonality Method for Multiplicative Model:</u>

It is generally considered to be best suited to forecasting time series that can be described by the equation:

$$y_t = (T_t * S_t * I_t)$$

This method is appropriate when a time series has a linear trend with a multiplicative seasonal pattern.

- Smoothing equation for the series
  $\mu_t = \alpha \frac{Yt}{St-p} + (1-\alpha)(\mu_{t-1} + b_{t-1}) \qquad 0 \leq \alpha \leq 1$
- Trend estimating equation
  $b_t = \beta(\mu_t - \mu_{t-1}) + (1-\beta)\, b_{t-1}$
- Seasonality updating equation
  $S_t = \gamma \frac{Yt}{\mu t} + (1-\gamma)\, S_{t-p}$
- Forecast equation
  $F_{t+m} = (\mu_t + m\, b_t)\, S_{t+m-p}$

where

μ_t → Exponentially smoothed value of the series at time t

y_t → Actual observation of time series at time t

T_t → Trend Estimate

α → Exponential Smoothing Constant for the data

β → Smoothing constant for trend

γ → Smoothing constant for seasonality

F_{t+m} → m period ahead forecasted value

p → the period of seasonality (p=4 for quarterly data & p=12 for monthly data

Initialising:

$$\mu_p = (y_1 + y_2 + \ldots .. y_p) / p$$

$$b_p = ((y_{p+1} + y_{p+2} + \ldots y_{p+p}) - (y_1 + y_2 + \ldots . y_p)) / p^2$$

$$S_i = y_i / \mu_p \qquad i=1,2,3\ldots.p$$

Choice of α,β,γ

α is used to smooth randomness, β to smooth trend and γ to smooth seasonality. Choose α,β,γ which minimize MSE or MAPE.

Holt- Winter's Trend and Seasonality Method for Additive Model:

It is generally considered to be best suited to forecasting time series that can be described by the equation:

$$y_t=(T_t + S_t + I_t)$$

- Exponentially smoothed series equation

  $\mu_t=\alpha\ (y_t-S_{t-p}) + (1-\alpha)\ (\mu_{t-1}+b_{t-1})$ $\quad\quad 0\leq \alpha \leq 1$

- Trend estimating equation

  $b_t = \beta(\mu_t - \mu_{t-1}) + (1-\beta)\ b_{t-1}$

- Seasonality updating equation

  $S_t = \gamma\ (y_t-\mu_t)+ (1-\gamma)\ S_{t-p}$

- Forecast equation

  $F_{t+m} = \mu_t + m\ b_t+S_{t+m-p}$

where

μ_t → Exponentially smoothed value of the series at time t

y_t → Actual observation of time series at time t

T_t → Trend Estimate

α → Exponential Smoothing Constant for the data

β → Smoothing constant for trend

γ → Smoothing constant for seasonality

F_{t+m} → m period ahead forecasted value

p → the period of seasonality (p=4 for quarterly data & p=12 for monthly data)

Initialising:

$\mu_p = (y_1 + y_2 + \ldots . y_p) / p$

$b_p = ((y_{p+1} + y_{p+2} + \ldots y_{p+p}) - (y_1 + y_2 + \ldots . y_p)) / p^2$

$S_i = y_i - \mu_p \qquad i=1,2,3\ldots.p$

Choice of α,β,γ

α is used to smooth randomness, β to smooth trend and γ to smooth seasonality. Choose α,β,γ which minimize MSE or MAPE.

- **Time series**- A time series consists of data which are arranged chronologically. It establishes a relationship between two variables in which one of the variable is independent variable i.e. the time and other variable y is the dependent variable whose value changes with regard to time variable e.g. total agricultural production in different years.

  Mathematically, a time series is defined by the values Y_1, Y_2, Y_3… Y_n of the variable Y at times t_1, t_2, t_3… t_n.

  Symbolically, Y = f (t), i.e., Y is a function of time t.

  Components of time series

  A time series consists of the following four components-

  - Trend
  - Seasonal variations
  - Cyclical variations
  - Irregular variations

- **Trend**-Trend refers to long term movement in the time series, i.e. Trend refers to the ability of the time series to increase or to decrease or to remain constant over a long period of time. If the values of the variable are scattered around a straight line, then we have a linear trend. Otherwise, the trend is non-linear e.g. long- term changes in productivity.

- **Seasonal variations**- Seasonal variations involve patterns of change within a year that tend to be repeated from year to year. They are short- term periodic movements. The time interval of occurrence of seasonal variations may vary from a few hours to a few weeks or a few months. To note the seasonal variations, the data must be recorded at least quarterly, monthly, weekly, or daily depending on the nature of the variable under consideration.

- **Cyclical variations**- Cyclical variations are oscillatory variations in the time series that oscillate around the trend line with period of oscillation as more than one year. These variations do not follow any regular pattern and move in somewhat unpredictable manner. These are upswings and downswings in the time series that are observable over extended periods of time.

- **<u>Irregular variations</u>**- The irregular component of the time series is the residual factor that accounts for the deviations of the actual time series values from what we would expect from the trend, seasonal, and cyclical components. It accounts for the random availability in the time series. The irregular component is caused by the short-term, unanticipated, and non-recurring factors that affect the time series, viz. earthquakes, floods etc.

<u>MODELS OF DECOMPOSITION</u>

There are two models of decomposition of time series:

- **<u>The additive model</u>**- This model is used when it is assumed that the four components are independent of one another, i.e., when the pattern of occurrence and the magnitude of movement in any particular component are not affected by other components under this assumption, the magnitude of time series (Y(t)), at any time t is the sum of the separate influences of its four components, i.e.

$$Y(t)= T(t) + S(t)+ C(t)+ I(t)$$

where T(t)= Trend variations
S(t)= seasonal variations
C(t)= cyclical variations
I(t)= irregular variations

- **<u>Multiplicative model</u>**-This model is used when it is assumed that the components may depend on each other.

$$Y(t)= T(t)*S(t)*C(t)*I(t)$$

MEASUREMENT OF TREND: Methods of measuring trend are as follows:

- **I.** Free hand method or graphical method
- **II.** Semi averages method
- **III.** Method of least squares
- **IV.** Moving averages method

- **Free hand method or Graphical method**- Original time series values are plotted for the values of Y(t) (on Y-axis) against t (on X-axis) to get an idea about the trend exhibited by the time series.

- **Semi-average method**-

Steps

i. Divide the series into two equal parts.
ii. Take average of each part separately.
iii. Plot the average of each part against the middle of the time period covered by the respective parts.
iv. Join the plotted points.

Note- If the number of time periods is even, we can divide such a data into two equal parts without ignoring any observation but if the number of time periods is odd, the normal practice is to ignore the middle period and divide the resulting series into two halves.

Q. Compute trend by the semi- average method of the following data:

Year	**Sales (lakhs of Rs.)**	**Semi- total**	**Semi- average**
1991	38		
1992	40		
1993	46	224	44.8
1994	49		

1995	51		
1996	55		
1997	61		
1998	63		
1999	69	345	69.0
2000	72		
2001	80		

These two semi- averages are plotted in the middle of the respective time spans. Thus 44.8 is plotted against 1993; and 69.0 against 1999. These two points are then connected by a straight line.

- **Method of least squares**

i. **Fitting a straight line(Line of best fit)-**

The equation of straight line is of the form **Y= a+bX**

By the method of least squares, the normal equations to find the values of a and b are

∑Y=na+b∑X

∑XY=a∑X+b∑(X^2)

Q. The following are the annual profits in thousands in a certain business. By the method of least squares fit a straight line.

Year	Profits(000)	t-1994 (X)	XY	X^2	Y_cap
1991	60	-3	-180	9	

1992	72	-2	-144	4	
1993	75	-1	-75	1	
1994	65	0	0	0	Y_cap=76+4.857*X
1995	80	1	80	1	
1996	85	2	170	4	
1997	95	3	285	9	
N=7	**∑Y =532**	**∑X=0**	**∑XY= 136**	**∑X^2= 28**	

The equation of straight line trend is Y_cap= a+b*X

By the method of least squares

a=∑Y/N= 76

b=∑X*Y/∑X^2=4.857

The trend equation would be **Y_cap= 76+ 4.857*X**

ii. <u>Fitting a quadratic trend</u>-

Y=a+bX+cX^2

In this case, the normal equations by the method of least squares are

∑Y=na+b∑X+c∑(X^2)

∑XY=a∑X+b∑(X^2)+c∑(X^3)

∑X^2*Y=a∑(X^2)+b∑(X^3)+c∑(X^4)

Q. Fit a quadratic trend on the following data.

X	**0**	**1**	**2**	**3**	**4**
Y	**1.0**	**1.5**	**1.5**	**2.3**	**3.5**

Sol.

X	**Y**	**X^2**	**X^3**	**X^4**	**X*Y**	**(X^2)*Y**
0	1	0	0	0	0	0
1	1.5	1	1	1	1.5	1.5
2	1.5	4	8	16	3	6.0
3	2.5	9	27	81	7.5	22.5
4	3.5	16	64	256	14	56.0
∑=10	**∑=10**	**∑=30**	**∑=100**	**∑=354**	**∑=26**	**∑=86**

By fitting the normal equations, we get

$10=5a+10b+30c$

$26=10a+30b+100c$

$86=30a+100b+354c$

Solving we get

$a=1.086$

$b=0.028$

$c=0.143$

Hence, the quadratic trend trend is given by

$$Y=1.086+0.028*X+0.143*(X^2)$$

iii. **<u>Exponential curves for trend values</u>-**

The equation of the exponential curves is of the following form:

$$y=a(b^x)$$

After taking the logarithm on both sides, the normal equations are

$$\sum \log y=n\log a+ \log b\sum x$$

$$\sum(x\log y)=\log a\sum x+\log b\sum(x^2)$$

OR

$$\sum Y = nA + B\sum x$$

$$\sum (xY) = A\sum x + B\sum (x^2)$$

Where, A = log a and B = log b

- **<u>Method of moving averages</u>**

This method measures trend by smoothing out the fluctuation of the data by means of moving average where a moving average of period 'm' is a series of successive averages of 'm' terms at a time by starting with 1st, 2nd, 3rdterm and so on. That means the 1staverage is the average of 1^{st} m terms, 2^{nd} average is the average of m terms starting from 2^{nd} to $(m+1)^{th}$ term , 3^{rd} average is the average of m terms starting from 3^{rd} to $(m+2)^{th}$ term and so on.

Case 1:

If m is odd i.e. m = 2n+1

Then the average is placed against the middle of time interval which it covers, i.e. t = n+1.

Case 2:

If m is even i.e. m = 2n

Then the average is placed between 2 middle values of the time interval which it covers i.e. t = n & t = n+1.

So in this case the moving average value doesn't coincide with the original time period and thus to make it coincide with the original time period we find centered moving average value by finding average of two periods at a time.

Q. Calculate the 5- yearly moving averages for the following data:

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Value('000rs)	123	140	110	98	104	133	95	105	150	135

Sol.

Year	Value('000rs)	5-yearly moving total("000rs)	5-yearly moving average('000rs)
1991	123	-	-
1992	140	-	-
1993	110	575	115
1994	98	585	117
1995	104	540	108
1996	133	535	107
1997	95	587	117.4
1998	105	618	123.6
1999	150	-	-
2000	135	-	-

MEASUREMENT OF SEASONAL VARIATIONS

The following methods are employed to measure seasonal variations:

Method of simple Averages

Steps involved:

(i) Find the quarterly (or seasonal) averages.

(ii) Find the average of quarterly (or seasonal) averages.

(iii) Express each quarterly (or seasonal) averages as a percentage of average of quarterly (or seasonal) averages. This gives seasonal indices.

Q. Assuming no trend in the series, calculate seasonal indices for the following data:

Year	**I quarter**	**II quarter**	**III quarter**	**IV quarter**
1994	**78**	**66**	**84**	**80**
1995	**76**	**74**	**82**	**78**
1996	**72**	**68**	**80**	**70**
1997	**74**	**70**	**84**	**74**
1998	**76**	**74**	**86**	**82**

Sol. Avg. of Quarterly averages = 76.4

	Quarter			
Year	I	II	III	IV
1994	78	66	84	80
1995	76	74	82	78
1996	72	68	80	70
1997	74	70	84	74
1998	76	74	86	82
Quarterly total	376	352	416	384
Quarterly Average	75.2	70.4	83.2	76.8
Seasonal indices	75.2/76.4*100 =98.43	70.4/76.4*100 =92.15	83.2/76.4*100 =108.90	76.8/76.4*100 =100.52

<u>Ratio to trend method</u>

Steps involved:

(i) Find the trend values with the help of method of least squares.

(ii) Divide the given original data (quarterly or monthly) by corresponding trend value and multiply this by 100 (i.e. they are trend eliminated). **The values so obtained are free from trend.**

(iii) Find the quarterly or monthly (as the case may be) averages of trend eliminated values.

(iv) Add the quarterly (or monthly) averages and the sum is say S and find constant factor by dividing 400 (or 1200) by S.

(v) Multiply each quarterly (or monthly) average obtained in step (iii) by the constant factor obtained in step (iv).

Q. For the given data below compute seasonal variations using ratio to trend method.

Year	**I**	**II**	**III**	**IV**
1996	**60**	**80**	**72**	**68**
1997	**68**	**104**	**100**	**88**
1998	**80**	**116**	**108**	**96**
1999	**108**	**152**	**136**	**124**
2000	**160**	**184**	**172**	**164**

Sol. First determine the trend on yearly basis and later we compute quarterly trend values.

Year X	**Yearly total**	**Quarterly average(Y)**	**Deviations from mid- value, i.e. 1998, x= X-1998**	**x*Y**	**x^2**	**Y_cap**
1996	280	70	-2	-140	4	64
1997	360	90	-1	-90	1	88
1998	400	100	0	0	0	112
1999	520	130	1	130	1	136
2000	680	170	2	340	4	160
N=5		**∑=560**	**0**	**∑=240**	**∑=10**	

a=∑Y/N=112; b=∑xY/∑(x^2)=24

Y_cap = 112 + 24*x = 112+24(X-1998)

Yearly increment=24; Quarterly increment=24/4=6

Calculation of quarterly trend values:

Consider 1996, trend value for middle quarter, i.e. half of 2nd and half of 3rd is 64. Quarterly increment is 6. So the trend value of 2nd quarter is 64-6/2=61 and for 3rd quarter is 64+6/2=67. Trend value for the first quarter is 61-6=55 and of the 4th quarter is 67+6=73.

Quarterly trend values

Year	**I**	**II**	**III**	**IV**
1996	55	61	67	73
1997	79	85	91	97
1998	103	109	115	121
1999	127	133	139	145
2000	151	157	163	169

The given values of the time series will now be expressed as percentages of the corresponding trend values given above. These are trend eliminated values.

Trend eliminated values

Year	**I**	**II**	**III**	**IV**
1996	109.09	131.15	107.46	93.15
1997	86.08	122.35	109.89	90.72
1998	77.67	106.42	93.91	79.34
1999	85.04	114.29	97.84	85.52
2000	105.96	117.20	105.52	97.04
Total	463.84	591.41	514.62	445.77
Quarterly average	92.77	118.28	102.92	89.15
Seasonal Indices	92.05	117.36	102.12	84.47

60/55*100=109.09, 80/61*100=131.15, etc.

Sum of the quarterly averages= 403.12

Constant factor=400/403.12=0.99226

Seasonal index for the first quarter=92.77*0.992226=92.05

Seasonal index for the second quarter=118.28*0.992226=117.36, and so on.

<u>Ratio to Moving Average (or percentage of Moving Average) method</u>

Steps involved:

(i) Take centered 12 monthly (or 4 quarterly) moving average values.

(ii) Express the original data as a percentage of the centered moving average values.

(iii) Arrange these percentage season wise for all the years. Average these percentages. These values are the preliminary seasonal indices.

(iv) Add these indices. If the sum is not 1200 or 400 for monthly or quarterly figures respectively. Then, multiply each value by the constant factor as explained below.

Constant factor = 1200/ sum of monthly indices or

= 400/sum of quarterly indices

This gives adjusted seasonal indices.

Q. Calculate seasonal indices by the ratio-to-moving average method from the following data:

Year	**I**	**II**	**III**	**IV**
1997	**68**	**62**	**63**	**78**
1998	**75**	**58**	**56**	**72**
1999	**60**	**63**	**67**	**93**
2000	**54**	**59**	**56**	**90**
2001	**59**	**55**	**58**	**65**

Sol.

Year	**Quarters**	**Barely prices(rs)**	**4-Figure moving total**	**2-figure moving total**	**4-figure moving average**	**Given figures as %of moving average Col.3/col.6*100**
1	**2**	**3**	**4**	**5**	**6**	**7**
1997	QI	68	-	-	-	-
	QII	62	-	-	-	-
	QIII	63	271	549	68.63	(63/68.83)*100=91.79
	QIV	78	278	552	69.00	(78/69)*100=113.04
1998	QI	75	274	541	67.63	(75/67.63)*100=110.90
	QII	58	267	528	66.00	(58/66)*100=87.88
	QIII	56	261	507	63.38	(56/63.38)*100=88.36
	QIV	72	246	497	62.13	(72/62.13)*100=115.89
1999	QI	60	251	513	64.13	(60/64.13)*100=93.56
	QII	63	262	545	68.13	(63/68.13)*100=92.47
	QIII	67	283	560	70.00	(67/70)*100=95.71
	QIV	93	277	550	68.75	(93/68.75)*100=135.27
2000	QI	54	273	535	66.88	(54/66.88)*100=80.74
	QII	59	262	521	65.13	(59/65.13)*100=90.59
	QIII	56	259	523	65.38	(56/65.38)*100=85.65
	QIV	90	264	524	65.50	(90/65.50)*100=137.50
2001	QI	59	260	522	65.25	(59/65.25)*100=90.42
	QII	55	262	499	62.38	(55/62.38)*100=88.17
	QIII	58	237			
	QIV	65	-			
			-			

Ratio to moving average values

Year	**I**	**II**	**III**	**IV**
1997	-	-	91.79	113.04
1998	110.90	87.88	88.36	115.89
1999	93.56	92.47	95.71	135.27
2000	80.74	90.59	85.65	137.50
2001	90.42	88.17	-	-
Total	375.62	359.11	361.51	501.17
Quarterly averages	93.9	89.77	90.373	125.3

Total of quarterly averages=399.345

Adjusted seasonal indices=93.9*400/399.345=94.05,

87.77*400/399.345=89.91

90.375*400/399.345=90.52,

125.3*400/399.345=125.50

Measurement of Cyclical Variation

We know that a time series consisting of annual data for longer periods is depicted by trend lines. This facilitates us to isolate the component of secular trend variation from the series and examine it for cyclical, seasonal and irregular components. Here, we will look at "Residual Method", by which one can isolate the cyclical variation component. This method can be bifurcated into two measures:

- Percent of Trend method
- Relative Cyclical Residual method.

Both these measures are expressed in terms of percentage. We look at each of them.

1. Percent of Trend Method:

When the ratio of actual values (Y) and the corresponding estimated trend values ($\hat{Y}$) is multiplied by 100, we are expressing the cyclical variation component as a percent of trend. Mathematically, we express it as

$$(Y/ \hat{Y}) * 100$$

2. Relative Cyclical Residual Method:

In this measure, we take the ratio of the difference between the Y and the corresponding $\hat{Y}$ values (that is, Y - $\hat{Y}$), and the $\hat{Y}$ values. To express these values in terms of percentage we multiply them by 100. In other words, the percentage deviation from the trend is found for all the values in the series. Mathematically, this is expressed as:

$$[(Y- \hat{Y})/ \hat{Y}] * 100$$

Example:

Year (t)	**Y**	**Y_e**	**$(Y/Y_e)*100$**	**$Y-Y_e$**	**$((Y-Y_e)/Y_e)*100$**
1989	77	83	92.77	-6	-7.22
1990	88	85	103.52	3	3.52
1991	94	87	108.04	7	8.04
1992	85	89	95.50	-4	-4.49
1993	91	91	100.00	0	0
1994	98	93	105.37	5	5.37
1995	90	95	94.73	-5	-5.26

In 1989, the percentage of trend indicates that the actual sales were 92.77% of the expected sales for that year.

For the same year, the relative cyclical residual indicates that the actual sales were 7.22% short of the expected value.

Methods to measure accuracy of the fitted model

- **Mean absolute error(MAE)**

$$MAE = Mean\ |e(t)|$$

- **Mean square error(MSE)**

$$MSE = Mean\ \{e(t)^2\}$$

Since, both these methods are scale dependent, we cannot use them to compare series which are on different scales.

For such purpose, we use **Mean absolute percentage error (MAPE)**

i.e. $Mean|p(t)|$

where, $p(t)=e(t)/y(t)*100$

STATIONARY RANDOM SERIES

A **strictly stationary stochastic process** is one where given $t_1, \ldots, t_n$; the joint statistical distribution of $X_{t1}, \ldots, X_{tn}$ is the same as the joint statistical distribution of $X_{t1+\tau}, \ldots, X_{t\ell+\tau}$ for all ℓ and τ.

This is an extremely strong definition: it means that all moments of all degrees (expectations, variances, third order and higher) of the process, anywhere are the same. It also means that the joint distribution of (X_t, X_s) is the same as (X_{t+r}, X_{s+r}) and hence cannot depend on s or t but only on the distance between s and t, i.e. $s - t$.

Since the definition of strict stationarity is generally too strict for everyday life a weaker definition of second order or weak stationarity is usually used.

Weak Stationarity or Covariance Stationarity means that mean and the variance of a stochastic process do not depend on t (that is they are constant) and the autocovariance between X_t and $X_{t+\tau}$ only can depend on the lag τ (τ is an integer, the quantities also need to be finite). Hence for stationary processes, $\{X_t\}$, the definition of autocovariance is

$\gamma(\tau) = \mathrm{cov}(X_t, X_{t+\tau})$, for integers τ.

AUTO COVARIANCE FUNCTION

Given a stochastic process $\{X_t\}$, the autocovariance is a function that gives the covariance of the process with itself at pairs of time points. If the process has the mean function $\mu_t = E[X_t]$, then the autocovariance is given by

$$C_{XX}(t,s) = cov(X_t, X_s) = E[(X_t - \mu_t)(X_s - \mu_s)] = E[X_t X_s] - \mu_t \mu_s.$$

Autocovariance is related to the more commonly used autocorrelation of the process in question.

In the case of a random vector $X = (X_1, X_2, \ldots, X_n)$, the autocovariance would be a square n X n matrix C_{XX} with entries $C_{XX}(j,k) = cov(X_j, X_k)$. This is commonly known as the covariance matrix or matrix of covariances of the given random vector.

AUTO CORRELATION FUNCTION

In general, the autocorrelation of a random process describes the correlation between values of the process at different times, as a function of the two times or of the time lag. Then X_i is the value (or realization) produced by a given run of the process at time *i*. Suppose that the process is further known to have defined values for mean μ_i and variance σ_i^2 for all times *i*. Then the definition of the autocorrelation between times *s* and *t* is

$$R(s,t) = \frac{\mathrm{E}[(X_t - \mu_t)(X_s - \mu_s)]}{\sigma_t \sigma_s},$$

where "E" is the expected value operator. Note that this expression is not well-defined for every time series or process, because the variance may be zero (for a constant process) or infinite. If the function *R* is well-defined, its value must lie in the range [−1, 1], with 1 indicating perfect correlation and −1 indicating perfect anti-correlation.

If X_t is a wide-sense stationary process, then the mean μ and the variance σ^2 are time-independent, and further the autocorrelation depends only on the lag between *t* and *s*: the correlation depends only on the time-distance between the pair of values but not on their position in time. This further implies that the autocorrelation can be expressed as a function of the time-lag, and that this would be an even function of the lag $\tau = s - t$. This gives the more familiar form

$$R(\tau) = \frac{\mathrm{E}[(X_t - \mu)(X_{t+\tau} - \mu)]}{\sigma^2},$$

and the fact that this is an even function can be stated as

$$R(\tau) = R(-\tau).$$

AUTO REGRESSIVE (AR) PROCESS

An auto regressive process of order p, i.e. AR (p) is a sequence of random variables $\{X_t\}$ characterised by

$X_t = \mu + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdot\cdot\cdot + \alpha_p X_{t-p} + e_t$, (11.16) where $\{e_t\}$ is a purely random process and represents the error term.

MOVING AVERAGE (MA) PROCESS:

A moving average process of order q, i.e. MA(q) is a sequence $\{X_t\}$ of random variables such that:

$$X_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

where μ is the mean of the series, the ϑ_1, ..., ϑ_q are the parameters of the model and the ε_t, ε_{t-1},..., ε_{t-q} are white noise error terms.

Thus, a moving-average model is conceptually a linear regression of the current value of the series against current and previous (unobserved) white noise error terms or random shocks. The random shocks at each point are assumed to be mutually independent and to come from the same distribution, typically a normal distribution, with location at zero and constant scale. Fitting the MA estimates is more complicated than with autoregressive models (AR models) because the lagged error terms are not observable. This means that iterative non-linear fitting procedures need to be used in place of linear least squares.

The moving-average model is essentially a finite impulse response filter applied to white noise, with some additional interpretation placed on it. The role of the random shocks in the MA model differs from their role in the AR model in two ways. First, they are propagated to future values of the time series directly: for example, ε_{t-1} appears directly on the right side of the equation for X_t. In contrast, in an AR model ε_{t-1} does not appear on the right side of the X_t equation, but it does appear on the right side of the X_{t-1} equation, and X_{t-1} appears on the right side of the X_t equation, giving only an indirect effect of ε_{t-1} on X_t. Second, in the MA model a shock affects X values only for the current period and *q* periods into the future; in contrast, in the AR model a shock affects X values infinitely far into the future, because ε_t affects X_t, which affects X_{t+1}, which affects X_{t+2}, and so on forever.

WHITE NOISE PROCESS

A white noise process is a random process of random variables {Xt} that are uncorrelated, have mean zero, and a finite variance (which is denoted s^2 below).

Formally, e_t is a white noise process if E $(e_t) = 0$, $E(e_t^2) = s^2$, and $E(e_t\, e_j) = 0$ for t not equal to j, where all those expectations are taken prior to times t and j.

AUTO REGRESSIVE MOVING AVERAGE (ARMA) PROCESS

Autoregressive–moving-average (ARMA) models provide a parsimonious description of a (weakly) stationary stochastic process in terms of two polynomials, one for the auto-regression and the second for the moving average.

Given a time series of data X_t, the ARMA model is a tool for understanding and, perhaps, predicting future values in this series. The model consists of two parts, an autoregressive (AR) part and a moving average (MA) part. The model is usually then referred to as the ARMA (*p*, *q*) model where *p* is the order of the autoregressive part and *q* is the order of the moving average part

The notation ARMA (*p*, *q*) refers to the model with *p* autoregressive terms and *q* moving-average terms. This model contains the AR (*p*) and MA (*q*) models,

$$X_t = c + \varepsilon_t + \sum_{i=1}^{p} \varphi_i X_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

The error terms ε_t are generally assumed to be independent and identically distributed random variables (i.i.d.) sampled from a normal distribution with zero mean: $\varepsilon_t \sim N(0, \sigma^2)$ where σ^2 is the variance. These assumptions may be weakened but doing so will change the properties of the model. In particular, a change to the i.i.d. assumption would make a rather fundamental difference.

ARMA (p, 0) → AR (p)

ARMA (0, q) → MA (q)

AUTO REGRESSIVE INTEGRATED MOVING AVERAGE (ARIMA) PROCESS

An **autoregressive integrated moving average (ARIMA)** model is a generalization of an autoregressive moving average (ARMA) model. These models are fitted to time series data either to better understand the data or to predict future points in the series (forecasting). They are applied in some cases where data show evidence of non-stationarity, i.e. when the mean & variance of some of the time series may be time variant.