GOAL PROGRAMMING

P. C. Jha

Introduction

- Goal programming is an approach used for solving a multi-objective optimization problem that balances a trade-off in conflicting objectives.
- It is an approach of deriving a best possible ‘satisfactory’ level of goal attainment.
- A problem is modelled into a goal programming model in a manner similar to that of a linear programming model. However, the goal programming model accommodates multiple and often conflicting incommensurable (dimension of goals and units of measurements may not be same) goals, in a particular priority order (hierarchy).
- A particular priority order is established by ranking or weighing various goals in accordance with their importance.
- The priority structure helps to deal with all goals that cannot be completely and/or simultaneously achieved, in such a manner that more important goals are achieved first, at the expense of the less important ones.

Concept

- Goal Programming can be thought of as an extension or generalization of linear programming to handle multiple, normally conflicting objective measures.
- Each of these measures is given a goal or target value to be achieved.
- Unwanted deviations from this set of target values are then minimized in an achievement function. This can be a vector or a weighted sum dependent on the goal programming variant used.
- As satisfaction of the target is deemed to satisfy the decision maker(s), an underlying satisficing philosophy is assumed.
- Goal programming is used to perform three types of analysis:
  - Determine the required resources to achieve a desired set of objectives.
  - Determine the degree of attainment of the goals with the available resources.
  - Providing the best satisfying solution under a varying amount of resources and priorities of the goals.

Terminology

- **Decision Maker:** The decision maker(s) refer to the person(s), organization(s), or stakeholder(s) to whom the decision problem under consideration belongs.
- **Decision Variable:** A decision variable is defined as a factor over which the decision maker has control. The set of decision variables fully describe the problem and form the decision to be made. The purpose of the goal programming model can be viewed as a search of all the possible combinations of decision variable values (known as decision space) in order to determine the point which best satisfies the decision maker's goals and constraints.
- **Criterion:** A criterion is a single measure by which the goodness of any solution to a decision problem can be measured. There are many possible criteria arising from different fields of application but some of the most commonly arising relate at the highest level to
  - Cost
  - Profit
  - Time
  - Distance
  - Performance of a system
  - Company or organizational strategy
  - Personal preferences of the decision maker(s)
  - Safety considerations
- A decision problem which has more than one criterion is therefore referred to as a multi-criteria decision making (MCDM) or multi-criteria decision aid (MCDA) problem. The space formed by the set of criteria is known as criteria space.

Terminology

- **Aspiration Level:** The numerical value specified by the decision maker that reflects his/her desire or satisfactory level with regard to the objective function under consideration. For example, suppose the company wishes to maximize the profit which is formulated as:

$$Max\ z = 2x_1 + 3x_2 \qquad \ldots(1)$$

Further suppose the management wishes to have at-least 40,000 as profit, then the above stated objective is required to be re-written as:

$$2x_1 + 3x_2 \geq 40{,}000 \qquad \ldots(2)$$

Here, 40,000 is the aspiration level with respect to profit.

- **Goal:** An objective function along with its aspiration level is called a goal. For example, the relation (1) is an objective function whereas relation (2) is a goal.

Terminology

- **Goal Deviation:** The difference between what we actually achieve and what we desire to achieve. There are two types of goal deviations:
  - Positive deviation or overachievement
  - Negative deviation or underachievement
- In general goals can be defined in three ways:
  - **Positive deviation:**
  $$f(x) \leq a$$
  $$f(x) - d^{+} = a$$
  - **Negative deviation:**
  $$f(x) \geq a$$
  $$f(x) + d^{-} = a$$
  - **Both deviations:**
  $$f(x) = a$$
  $$f(x) - d^{+} + d^{-} = a$$
- **Remark:** In general,, for goal programming irrespective of the type of the goal we can use both the deviations for each case. However, for the first two cases it is required to minimize just one of the deviation only.

Formulation

- Desirable vs. Undesirable Deviations: (depend on the objectives)
  - Max goals (≥) - the more the better - d_i^+or p_i desirable.
  - Min goals (≤) - the less the better - d_i^-or n_i desirable.
  - Exact goals (=) - exactly equal - both d_i^+ (or p_i) and d_i^-(or n_i) undesirable
- In all the situations, we first identify the undesirable deviation of the expression in the goal and then attempt to minimize the same.
- In GP, the objective is to minimize the (weighted) sum of undesirable deviations (all undesirable d_i^+ (or p_i) and d_i^-(or n_i)→→ 0).
- For each goal, at least, one of d_i^+ (or p_i) and d_i^-(or n_i) must be equal to “0”.
- An optimal solution is attained when all the goals are reached as close as possible to their aspiration level, while satisfying a set of constraints.

Types

There are two types of goal programming formulations:

- **Non Pre-emptive Goal Programming:** In this type of problem we try to minimize the weighted sum of all the undesirable deviations. That is in this type no goal is said to dominate any other goal. However, it is possible to have different importance for the deviations by the decision makers. For example, Let us consider the following multi-objective linear programming problem (MOP_1):

$$Max\ (Profit)\ z_1 = 2x_1 + 3x_2$$
$$Min\ (Cost)\ z_2 = x_1 + 5x_2$$
$$subject\ to,$$
$$x_1 + x_2 \leq 10$$
$$x_1 - x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Types

- The above problem can be converted into a goal programming problem assuming that the decision maker wishes to have at-least 40,000 profit and the cost should not exceed the limit of 20,000 represented as follows (GP_1):

$$Min\ d_1^- + d_2^+$$
$$subject\ to,$$
$$2x_1 + 3x_2 + d_1^- = 40{,}000$$
$$x_1 + 5x_2 - d_2^+ = 20{,}000$$
$$x_1 + x_2 \leq 10$$
$$x_1 - x_2 \leq 4$$
$$x_1, x_2 \geq 0$$
$$d_1^-, d_2^+ \geq 0$$

- The above is the representation of non pre-emptive goal programming problem.

Types

There are two types of goal programming formulations:

- **Pre-emptive Goal Programming:** Suppose in the above problem after knowing the fact that the multi-objective scenario restrict to have any such solution which satisfies both the goals simultaneously, then the decision makers specifies the priorities for both the goals. Suppose in problem GP_1 the first goal is having the higher priority, say P_1, and the second goal is having lower priority, say P_2, that is $P_1 > P_2$. In this situation, the problem GP_1 is written as follows (GP_2):

$$Min\ \{P_1 d_1^-, P_2 d_2^+\}$$
$$subject\ to,$$

$$2x_1 + 3x_2 + d_1^- = 40{,}000$$
$$x_1 + 5x_2 - d_2^+ = 20{,}000$$
$$x_1 + x_2 \leq 10$$
$$x_1 - x_2 \leq 4$$
$$x_1, x_2 \geq 0$$
$$d_1^-, d_2^+ \geq 0$$
$$P_1 > P_2$$

- The above is the representation of pre-emptive goal programming problem.

Note

- There are two types of constraints in a goal programming problem: soft constraints and hard (or rigid) constraints.
- The soft constraints are the constraints corresponding to the goals which has been obtained by using the aspirations for the objective functions. For example the first two constraints in the above problems (GP_1, GP_2) are soft constraints.
- Hard constraints are the constraints corresponding to the feasible region or the original constraints in which no violation is acceptable. For example the constraints in problem MOP_1 are hard constraints in the above problems (GP_1, GP_2).

Example

Alpha company produces two kinds of fancy products, pen holder and paper tray. Production of either of them requires 1 hr production capacity in the plant. The plant has a maximum production capacity of 12 hrs per week. The maximum number of pen holders and paper trays that can be sold are 7 and 10 respectively. The gross margin from the sales of pen holder is Rs 90 and Rs 45 for a paper tray. The overtime hours should not exceed 3 hrs per week if required. The plant manager has set the following goals in order of importance:

- P_1: He wants to avoid any under-utilization of production capacity
- P_2: He wants to limit the overtime hours to 3 hrs
- P_3: He wants to sell as many pen holders and paper trays as possible. Since the gross margin from the sale of a pen holder is set at twice the amount of the profit from a paper tray, the manager has twice as much desire to achieve the sales goal for pen holders as for paper trays.
- P_4: The manager wishes to minimize the overtime operation of the plant as much as possible.

Formulation

- Let x_1 be the number of pen holders to be produced per week and x_2 be the number of paper trays to be produced per week, then the above problem can be formulated as:

$$Min\ \{P_1 d_1^-, P_2 d_2^+, P_3(2d_3^- + d_4^-), P_4 d_1^+\}$$

$$subject\ to$$

$$x_1 + x_2 + d_1^- - d_1^+ = 12$$

$$d_1^+ - d_2^+ = 3$$

$$x_1 + d_3^- = 7$$

$$x_2 + d_4^- = 10$$

$$x_1, x_2, d_1^-, d_1^+, d_2^+, d_3^-, d_4^- \geq 0$$

Problem

Harrison Electric produces two products popular with home renovators, old fashioned chandeliers and ceiling fans. Both chandeliers and fans require a two-step production process involving wiring and assembly. It takes about 2 hrs to wire a chandelier and 3 hrs to wire a fan. Final assembly of the chandelier and fan require 6 and 5 hrs respectively. The production capability is such that only 12 hrs of wiring and 30 hrs of assembly time are available. Each chandelier produced nets the firm $7 and each fan $6. The Harrison's management wants to achieve the following goals with the given priorities:

- P_1: Reach a profit as much above $30 as possible.
- P_2: Fully use wiring department hours available.
- P_3: Avoid assembly department overtime.
- P_4: Produce at-least 7 ceiling fans.

Formulate and solve the above goal programming problem using graphical method.

Formulation

Let x_1 be the number of chandeliers to be produced per week and x_2 be the number of ceiling fans to be produced per week, then the above problem can be formulated as:

$$Min\ \{P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^-\}$$

Subject to,

$$7x_1 + 6x_2 + d_1^- - d_1^+ = 30 \quad \text{(Profit)}$$

$$2x_1 + 3x_2 + d_2^- - d_2^+ = 12 \quad \text{(Wiring)}$$

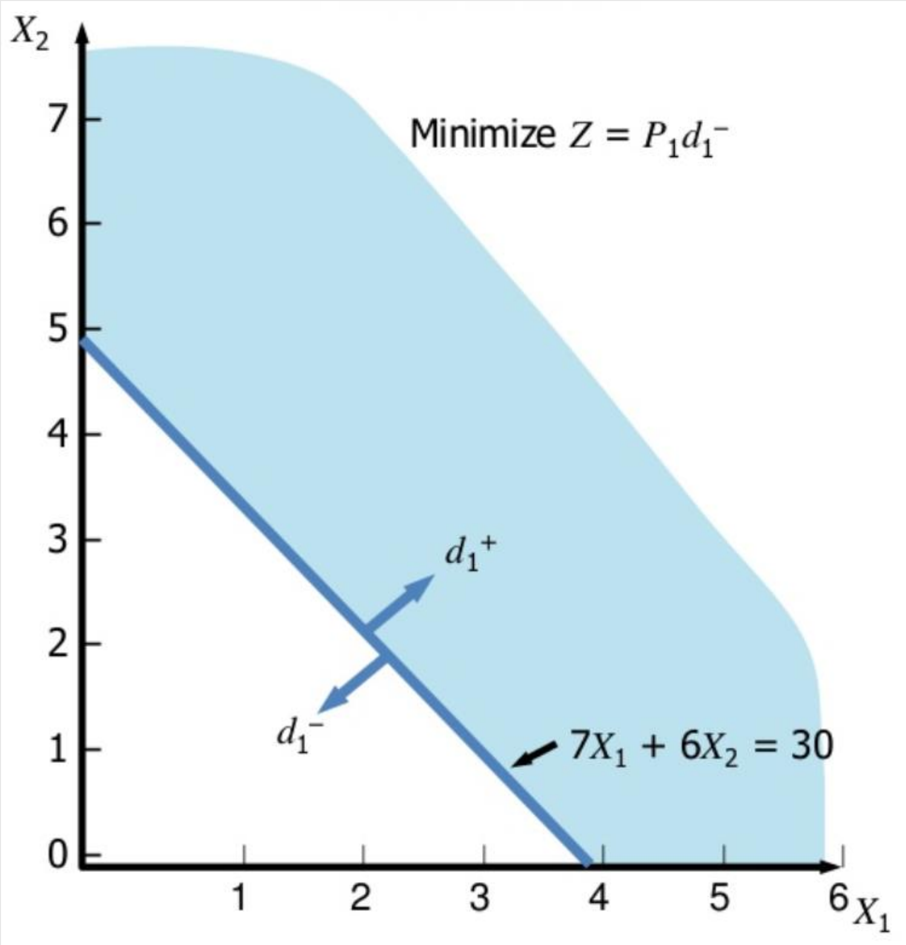
$$6x_1 + 5x_2 + d_3^- - d_3^+ = 30 \quad \text{(Assembly)}$$

$$x_2 + d_4^- - d_4^+ = 7 \quad \text{(Ceiling Fan)}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ \quad \text{(Non-negativity)}$$

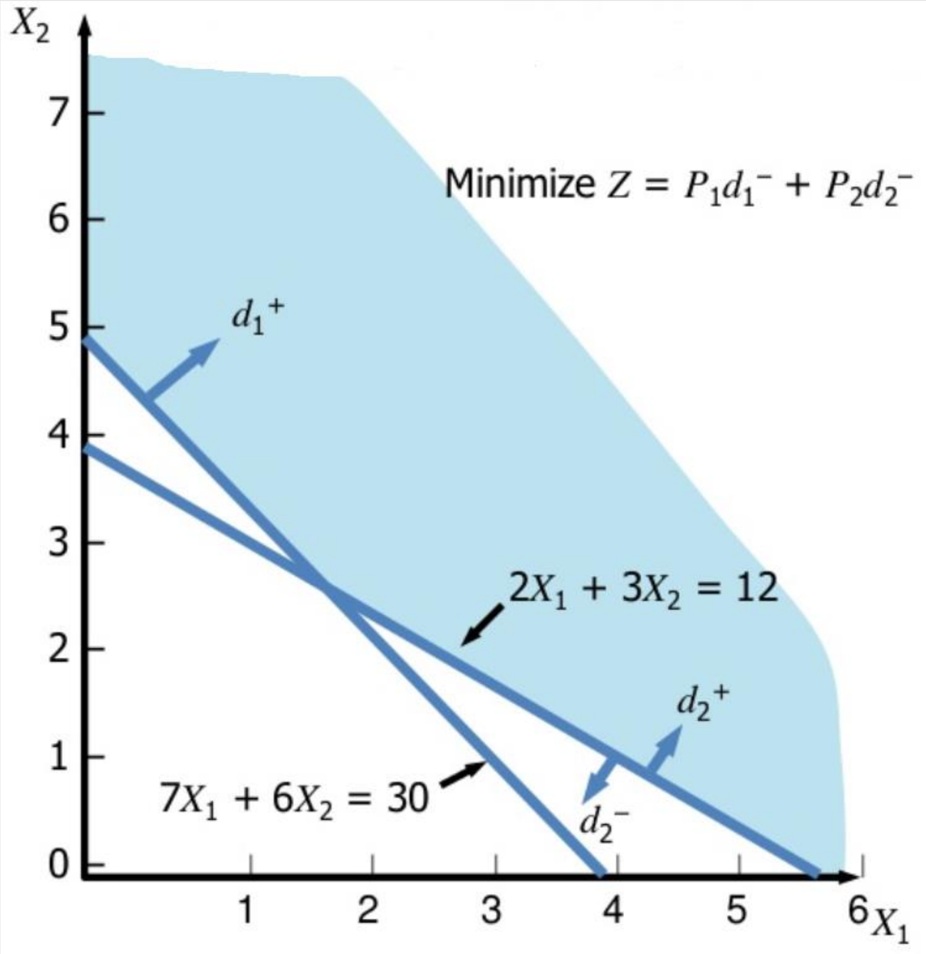
Graphical Method

- To solve this we graph one constraint at a time starting with the constraint having the highest priority.
- In this case we start with the profit constraint as it has the variable d_1^- with highest priority P_1.
- Note that in graphing these constraints, the deviational variables are ignored.
- To minimize d_1^-, the shaded region is the feasible region.



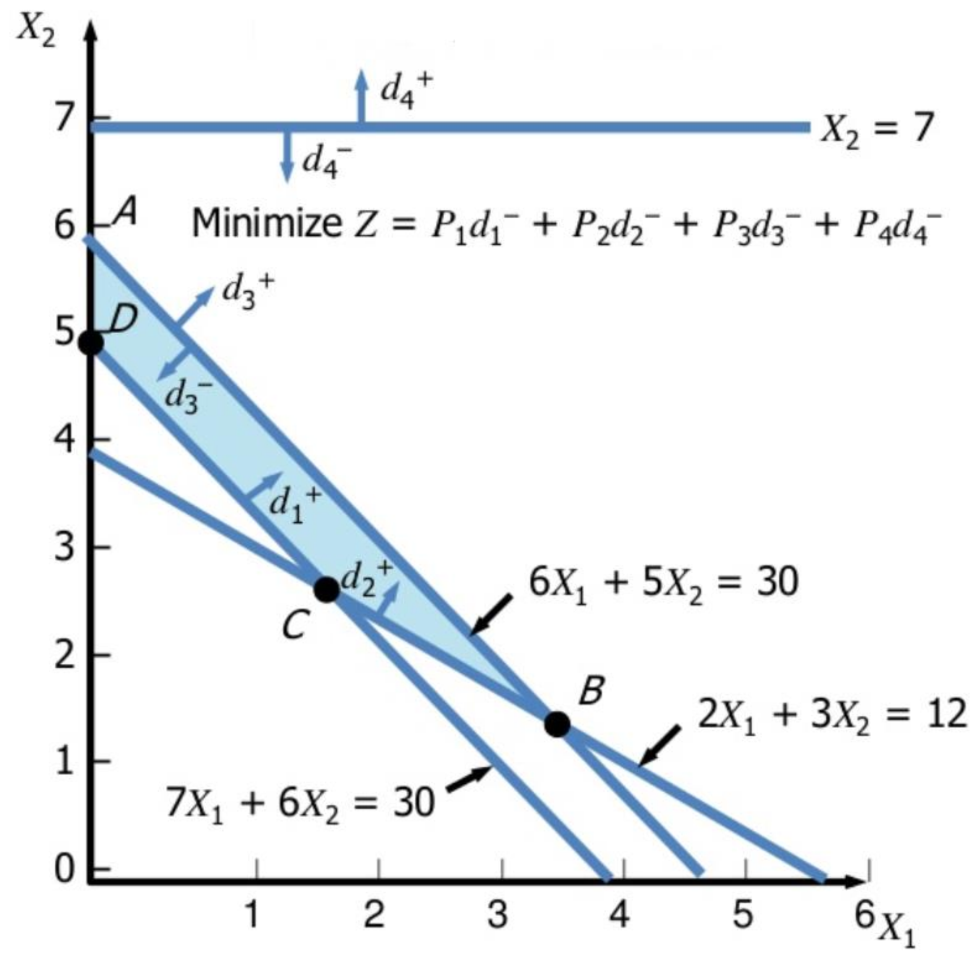
Graphical Method

- The next step is to plot the second priority goal of minimizing d_2^-.
- The region below the constraint line $2x_1 + 3x_2 = 12$ represents the values for d_2^- while the region above the line stands for d_2^+.
- To avoid under-utilizing the available hours the area under the line is avoided.
- The graph represents the common feasible region of both the goals.



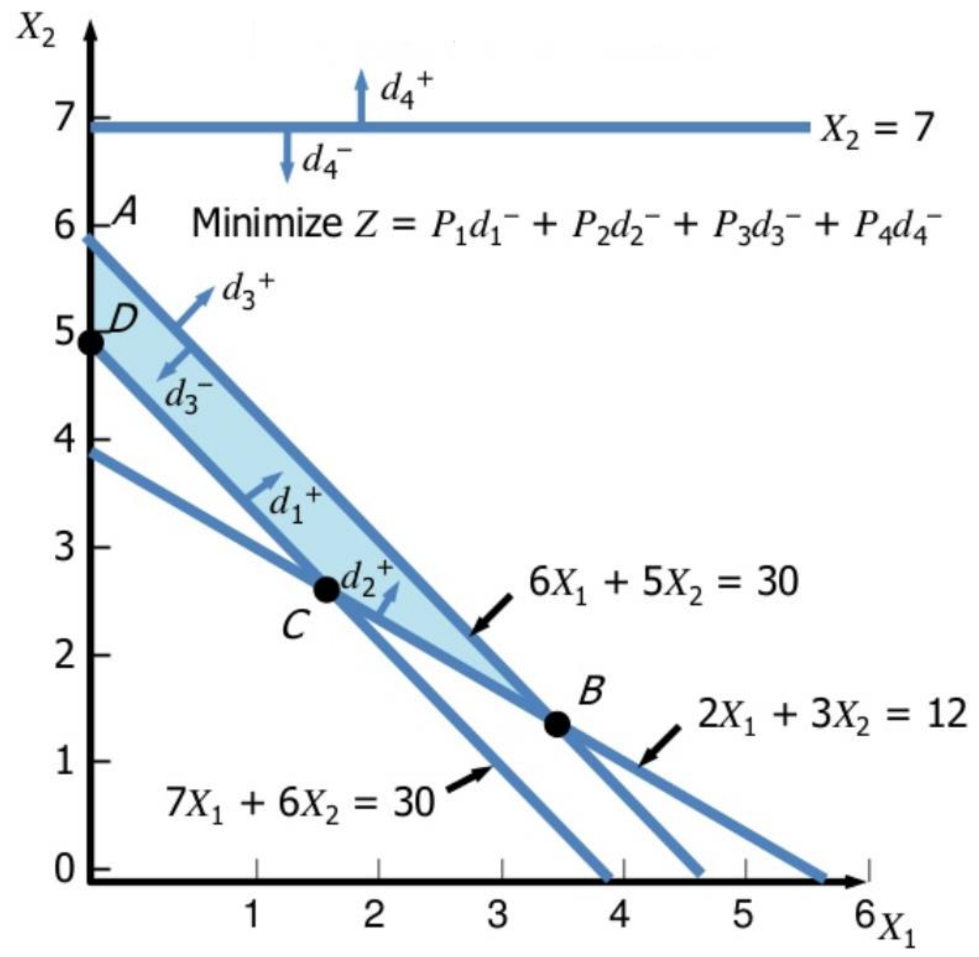
Graphical Method

- The third goal is to avoid over-time of the assembly hours. So we want d_3^+ to be as close to zero as possible.
- This goal can be obtained as shown in the figure as it has common feasible region with the previous two goals.
- The fourth goal seeks to minimize d_4^-.
- To do this requires eliminating the area below the constraint line $x_2 = 7$ which is not possible as the previous goals are of higher priority.



Graphical Method

- The optimal solution must satisfy the first three goals and come as close as possible to satisfying the fourth goal.
- This would be point A on the graph with $x_1 = 0$ and $x_2 = 6$.
- Substituting into constraints we find $d_1^- = 0, d_1^+ = 6, d_2^- = 0, d_2^+ = 6, d_3^- = 0, d_3^+ = 0, d_4^- = 1, d_4^+ = 0$
- A profit of \$36 is achieved, exceeding the goal.



Note

- The graphical solution procedure can also be worked out as mentioned:

1. Rank the goals of the problem in order of importance (i.e., priority wise).
2. Identify the feasible solution points that satisfy the problem constraints.
3. The solution procedure considers one goal at a time, starting with the highest priority goal and ending with the lowest. ***The process is carried out such that the solution obtained from a lower – priority goal never degrades any higher – priority solutions.*** Identify all feasible solutions that achieve the highest – priority goal; if no feasible solutions achieve the highest – priority goal, identify the solution(s) that comes closest to achieving it. Let the highest priority goal, G1, attain a value G1 = G1*.
4. Move down one priority level. Add the constraint G1 = G1* to the existing constraints of the problem and determine the "best" solution.
5. Repeat Step 4 until all priority levels have been considered.

- The following example illustrates this procedure.

Problem

A client has $80,000 to invest and, as an initial strategy, would like the investment portfolio restricted to two stocks:

Stock	Price/Share	Estimated Annual Return / Share	Risk Index / Share
U. S. Oil	$25	$3	0.50
Hub Properties	$50	$5	0.25

U. S. Oil, which has a return of $3 on a $25 share price, provides an annual rate of return of 12%, whereas Hub Properties provides an annual rate of return of 10%. The risk index per share, 0.50 for U. S. Oil and 0.25 for Hub Properties, is a rating Nicolo assigned to measure the relative risk of the two investments. Higher risk index values imply greater risk; hence, Nicolo judged U. S. Oil to be the riskier investment. By specifying a maximum portfolio risk index, Nicolo will avoid placing too much of the portfolio in high risk investments.

To illustrate how to use the risk index per share to measure the total portfolio risk, suppose that Nicolo chooses a portfolio that invests all $80,000 in U. S. Oil, the higher risk but higher return, investment. Nicolo could purchase $80,000/$25 = 3200 shares of U. S. Oil, and the portfolio would have a risk index of 3200(0.50) = 1600. Conversely, if Nicolo purchases no shares of either stock, the portfolio will have no risk, but also no return. Thus, the portfolio risk index would vary from 0 (least risk) to 1600 (most risk).

Nicolo's client would like to avoid a high risk portfolio; thus, investing all funds in U. S. Oil would not be desirable. However, the client agreed that an acceptable level of risk would correspond to portfolios with a maximum total risk index of 700 or less.

Another goal of the client is to obtain an annual return of at least $9000. This goal can be achieved with a portfolio consisting of 2000 shares of U. S. Oil [at a cost of 2000($25) = $50,000] and 600 shares of Hub Properties [at a cost of 600($50) = $30,000]; the annual return in this case would be 2000($3) + 600($5) = $9,000. Note, however, that the portfolio risk index for this investment strategy would be 2000(0.50) + 600(0.25) = 1150; thus, this portfolio achieves the annual return goal but does not satisfy the portfolio risk index goal.

Suppose that the client's top – priority goal is to restrict the risk; that is, keeping the portfolio risk index at 700 or less is so important that the client is not willing to trade the achievement of this goal for any amount of an increase in annual return. As long as the portfolio risk index does not exceed 700, the client seeks the best possible return. Based on this statement of priorities, the goals for the problem are as follows:

Primary Goal (Priority Level 1)
Goal 1: Find a portfolio that has a risk index of 700 or less.

Secondary Goal (Priority Level 2)
Goal 2: Find a portfolio that will provide an annual return of at least $9,000

Problem Formulation

Lex Min z = $\{P_1\ d_1^+,\ P_2\ d_2^-\}$

d^- : negative deviational variable
d^+: positive deviational variable

subject to:

$$25U + 50H \leq 80{,}000 \quad \text{(Funds available)}$$
$$0.50U + 0.25H - d_1^+ + d_1^- = 700 \quad \text{(Goal 1)}$$
$$3U + 5H - d_2^+ + d_2^- = 9{,}000 \quad \text{(Goal 2)}$$

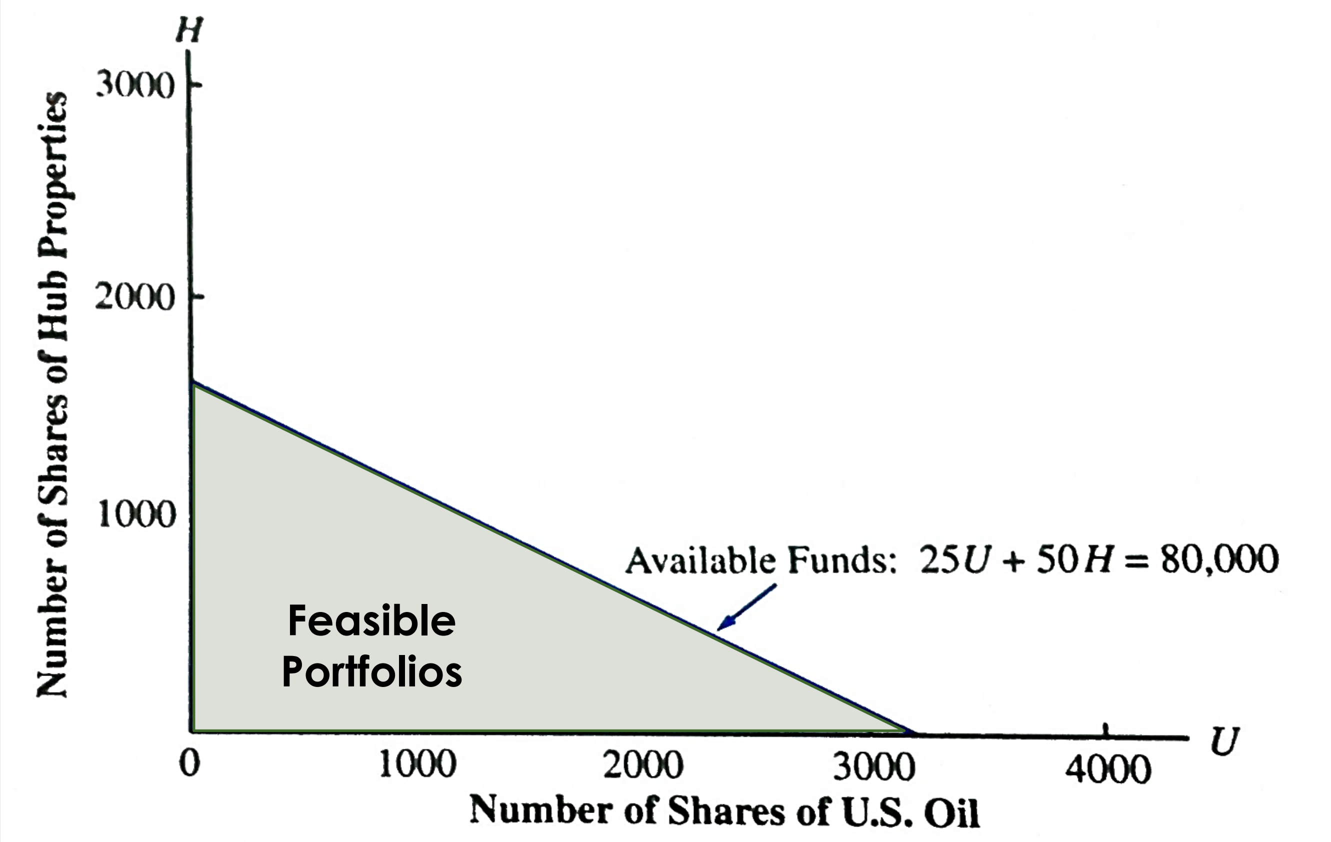
$$U, H, d_1^+, d_1^-, d_2^+, d_2^- \geq 0$$

where:

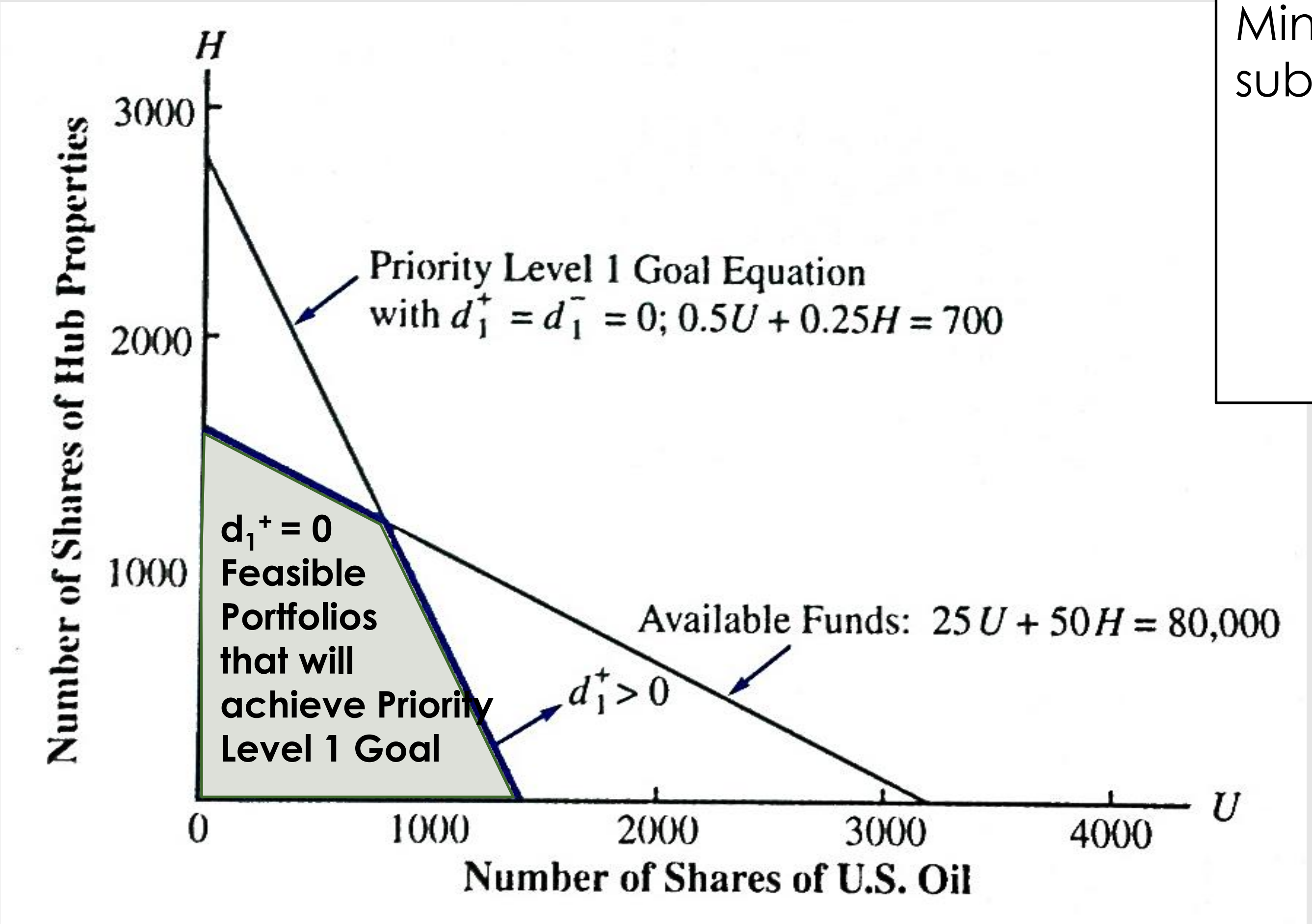
U: number of shares of U. S. Oil purchased
H: number of shares of Hub Properties purchased

Portfolios that satisfy the Available Funds Constraint

25U	+	50H	≤	80,000
		(Funds available)		
		U, H	≥	0



Portfolios that satisfy the P1 Goal



Min d_1^+

subject to:

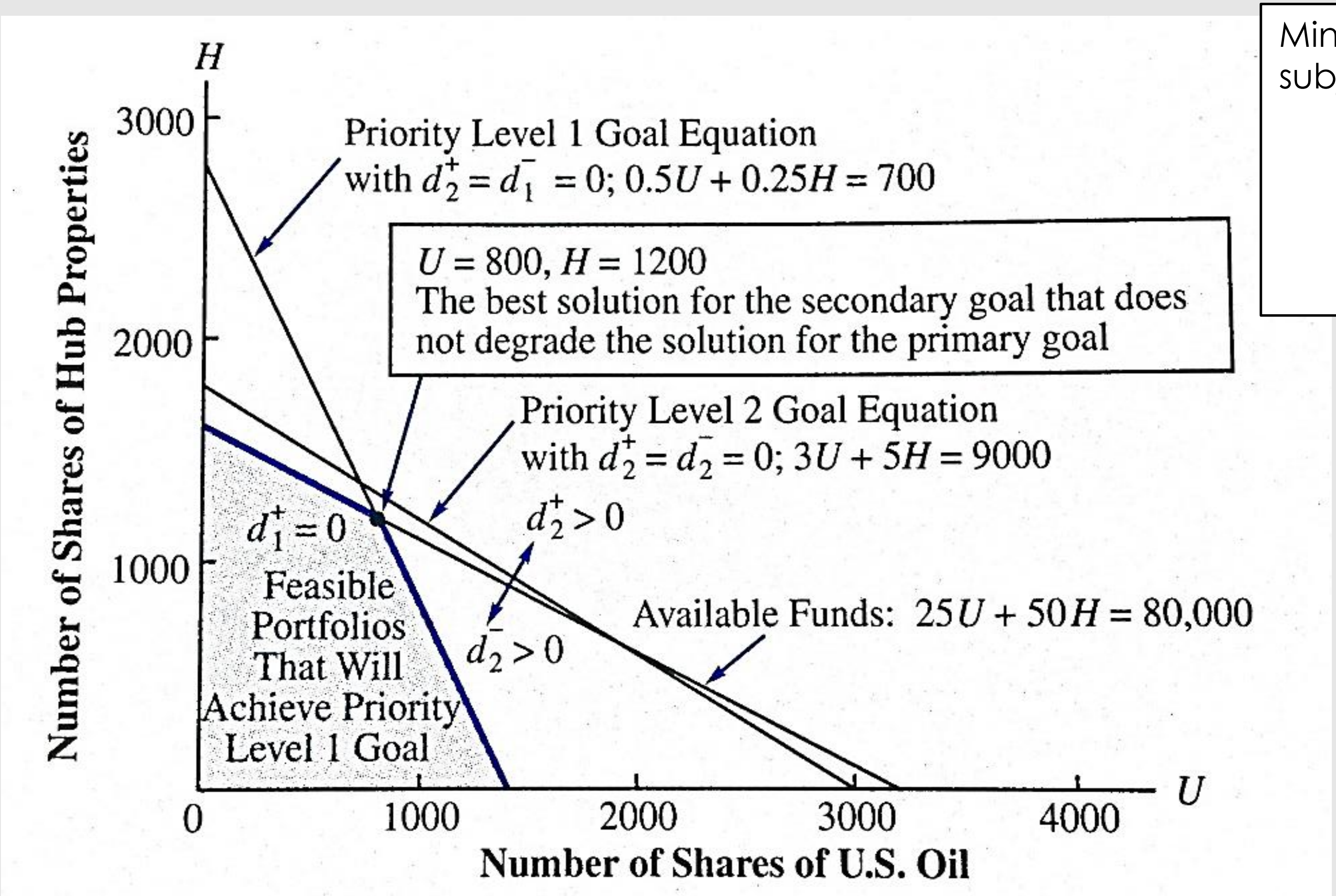
$$25U + 50H \le 80{,}000$$

$$0.50U + 0.25H - d_1^+ + d_1^- = 700$$

$$3U + 5H - d_2^+ + d_2^- = 9{,}000$$

$$U, H, d_1^+, d_1^-, d_2^+, d_2^- \ge 0$$

Best Solution with respect to Both Goals



Min d_2^-

subject to:

$$
\begin{aligned}
25U + 50H &\le 80,000 \\
0.50U + 0.25H - d_1^+ + d_1^- &= 700 \\
3U + 5H - d_2^+ + d_2^- &= 9,000 \\
d_1^+ &= 0 \\
U, H, d_1^+, d_1^-, d_2^+, d_2^- &\ge 0
\end{aligned}
$$

Final Solution

Thus, the solution recommends that :

U = 800 shares
H = 1200 shares

Note that the priority level 1 goal of a portfolio risk index of 700 or less has been achieved. However, the priority level 2 goal of at least a \$9,000 annual return is not achievable. The annual return for the recommended portfolio is \$8,400.

Problem

A textile company produces two types of materials A and B. Material A is produced according to direct orders from furniture manufacturers. The material B is distributed to retail fabric stores. The average production rates for material A and B are identical at 1000 metres/hour. By running two shifts the operational capacity of the plant is 80 hours per week. The marketing department reports that the maximum estimated sales for the following week is 70000 metres of material A and 45000 metres of material B. According to the accounting department the profit from a metre of material A is Rs. 2.50 and from a metre of material B is Rs. 1.50. The management of the company decides that a stable employment level is the primary goal for the firm. Therefore, whenever there is demand exceeding normal production capacity, management simply expands production capacity by providing overtime. However, management feels that overtime operation of the plant of more than 10 hours per week should be avoided because of the accelerating costs. The management has the following goals in the order of importance:

- The first goal is to avoid any under-utilization of production capacity.
- The second goal is to limit the overtime operation of the plant to 10 hours.
- The third goal is to achieve the sales goals of 70000 and 45000 respectively for both the materials.
- The last goal is to minimize the overtime operation of the plant as much as possible.

Formulate this as a goal programming problem to help the management for the best decision and solve the problem using simplex method.

Formulation

- Let x_1 be the number of hours used for producing material A per week and x_2 be the number of hours used for producing material B per week, then the above problem can be formulated as:

$$Min\ z = \{P_1 d_1^-, P_2 d_{12}^+, 5P_3 d_2^-, 3P_3 d_3^-, P_4 d_1^+\}$$

$$x_1 + x_2 + d_1^- - d_1^+ = 80 \quad \text{(Production Capacity Constraint)}$$

$$x_1 + d_2^- = 70 \quad \text{(Sales Constraint for Material A)}$$

$$x_2 + d_3^- = 45 \quad \text{(Sales Constraint for Material B)}$$

$$d_1^+ + d_{12}^- - d_{12}^+ = 10 \quad \text{(Overtime Operation Constraint)}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_3^-, d_{12}^-, d_{12}^+ \geq 0 \quad \text{(Non-negativity restriction)}$$

Simplex Method

Before the solution by the simplex method is presented for the goal programming problem, a few points to be observed are given below:

- In goal programming the purpose is to minimize the unattained portion of the goal as much as possible. This is achieved by minimizing the deviational variables.
- It should be remembered that pre-emptive priority factors are ordinal weights and they are not commensurable. Consequently, Z_j or $(Z_j - C_j)$ cannot be expressed by a single row as in linear programming. Rather, the simplex criterion becomes a matrix of (*m* x *n*) size, where *m* represents the number of pre-emptive factors and *n* is the number of variables.
- Since the simplex criterion $(Z_j - C_j)$ is expressed as a matrix rather than a row, a new procedure must be devised for identifying the key column. Again since P_j >> P_{j+1}, the selection procedure of the column must be initiated from P_j and move gradually to the lower priority levels.

Simplex Method

The initial simplex table of the given problem is

Cj			0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2
C_B	X_B	b	x_1	x_2	d_1^-	d_2^-	d_3^-	d_{12}^-	d_1^+	d_{12}^+
P_t	d_1^-	80	1.	1	1	0	0	0	-1	0
$5P_3$	d_2^-	70	1	0	0	1	0	0	0	0
$3P_3$	d_3^-	45	0	1	0	0	1	0	0	0
0	d_{12}^-	10	0	0	0	0	0	1	1	-1
	P_4	0	0	0	0	0	0	0	-1	0
Zj-Cj	P_3	485	5	3	0	0	0	0	0	0
	P_2	0	0	0	0	0	0	0	0	-1
	P_1	80	1	1	0	0	0	0'	-1	0

The first four rows of the table are set up in the same way as for linear programming with the coefficients of the associated variables placed in the appropriate entries. Below the thick line which separates the constraints from the objective function, there are four rows and each row stands for a priority goal level.

Simplex Method

The values of $(Z_j - C_j)$ are computed as follows:

$$Z_j - C_j = (Elements\ in\ C_B\ column) * (Corresponding\ elements\ in\ X\ columns) - C_j$$

For example, for column x_1

$$Z_1 - C_1 = P_1 * 1 + 5P_3 * 1 + 3P_3 * 0 + 0 * 1 - 0 = P_1 + 5P_3$$

Similarly, for column x_2

$Z_2 - C_2 = P_1 + 3P_3$ and similarly for other columns.

Since, P_1, P_2, P_3 and P_4 are not commensurable, we must list their coefficients separately in their rows in the simplex criterion $(Z_j - C_j)$ as shown in the above table.

Simplex Method

It should be apparent that the selection of the key column is based on the per unit contribution rate of each variable in achieving the goals. When the first goal is completely attained, then the key column selection criterion moves on to the second goal and so on. This is why the pre-emptive factors are listed from the lowest to the highest so that the key column can be easily identified at the bottom of the table.

In goal programming the Z_i values in the resources column (X_B) represents the unattained portion of the goal.

The key column would be determined by selecting the largest positive element in $Z_j - C_j$ row at the P_1 level as there exists an unattained portion of this highest goal. There are two identical positive values in the X_1 and X_2 columns. In order to break this tie we check the next lower priority levels. Since at priority 3 (P_3), the largest element is 5 in a row, therefore, X_1 becomes the key column.

Simplex Method

The key row is determined by selecting the minimum positive or zero value when values in the resources column (X_B) are divided by the coefficients in the key column.

In the given table d_2^- is the key row.

By utilizing the usual simplex procedure the previous table is updated in the table given below

C_j			0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2
C_B	X_B	b	x_1	x_2	d_1^-	d_2^-	d_3^-	d_{12}^-	d_1^+	d_{12}^+
P_1	d_1^-	10	0	1	1	−1	0	0	−1	0
0	x_1	70	1	0	0	1	0	0	0	0
$3P_3$	d_3^-	45	0	1	0	0	1	0	0	0
0	d_{12}^-	10	0	0	0	0	0	1	1	−1
	P_4	0	0	0	0	0	0	0	−1	0
$Z_j - C_j$	P_3	135	0	3	0	−5	0	0	0	0
	P_2	0	0	0	0	0	0	0	0	−1
	P_1	10	0	1	0	−1	0	0	−1	0

Simplex Method

Again, the above table does not give the optimal solution as the resources column indicates unattained portion of goals. Proceeding in the same manner as above, an improved solution can be obtained if d_1^- is driven out and decision variable x_2 enters into the solution. The new improved solution is shown in the table given below

C_j			0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2
C_B	X_B	b	x_1	x_2	d_1^-	d_2^-	d_3^-	d_{12}^-	d_1^+	d_{12}^+
0	x_2	10	0	1	1	−1	0	0	−1	0
0	x_1	70	1	0	0	1	0	0	0	0
$3P_3$	d_3^-	35	0	0	−1	1	1	0	1	0
0	d_{12}^-	10	0	0	0	0	0	1	1	−1
Z_j-C_j	P_4	0	0	0	0	0	0	0	-1	0
	P_3	105	0	0	−3	−2	0	0	3	0
	P_2	0	0	0	0	0	0	0	0	−1
	P_1	0	0	0	−1	0	0	0	0	0

Simplex Method

The solution in the above table indicates production of 70000 metres of material A and 10000 metres of material B is sufficient to achieve the first, second and fourth goals and the value of $d_3 = 35$ suggests that 35000 metres of material B is not achieved.

It is also observed that all the elements in P_1 and P_2 are either zero or negative which indicates that the first two priorities are achieved. Therefore, to improve the solution, the selection of the key column is done at P_3 level. Since, the only positive element 3 occurs at P_3 level which lies in d_1^+ column, thus d_1^+ enters into the solution and d_{12}^- is driven out as shown in the table below

C_j			0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2
C_B	X_B	b	x_1	x_2	d_1^-	d_2^-	d_3^-	d_{12}^-	d_1^+	d_{12}^+
0	x	20	0	1	1	−1	0	1	0	−1
0	x_1	70	1	0	0	1	0	0	0	0
$3P_3$	d_3^-	25	0	0	−1	1	1	−1	0	1
P_4	d_1^+	10	0	0	0	0	0	1	1	−1
	P_4	10	0	0	0	0	0	1	0	−1
$Z_j - C_j$	P_3	75	0	0	−3	−2	0	−3	0	3
	P_2	0	0	0	0	0	0	0	0	−1
	P_1	0	0	1	−1	0	0	0	0	0

Simplex Method

In the above table, since the third goal is not completely attained, there is a positive value in $(Z_j - C_j)$ at the P_3 level. We find it in the d_{12}^+ column. Obviously, we can attain the third goal to a greater extent if we introduce d_{12}^+ in the solution. We find, however a negative value at the higher priority level that is at P_2. This implies if we introduce d_{12}^+ we would improve the achievement of third goal but at the expense of achieving the second goal. Thus, we cannot introduce d_{12}^+. Similarly, for d_{12}^-.

Thus, the above table presents the optimal solution.

The optimal solution is $x_1 = 70, x_2 = 20, d_1^+ = 10, d_3^- = 25$. In other words, the company should produce 70000 metres of material A and 20000 metres of material B with 10 hours of overtime of the plants resulting in 25000 metres of under-achievement in the sales goal of material B.

Harrison Electric Problem using Simplex Method

- Recall the Harrison Electric model given below:

Let x_1 be the number of chandeliers to be produced per week and x_2 be the number of ceiling fans to be produced per week, then the above problem can be formulated as:

$$Min\ \{P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^-\}$$

Subject to,

$$7x_1 + 6x_2 + d_1^- - d_1^+ = 30 \quad \text{(Profit)}$$

$$2x_1 + 3x_2 + d_2^- - d_2^+ = 12 \quad \text{(Wiring)}$$

$$6x_1 + 5x_2 + d_3^- - d_3^+ = 30 \quad \text{(Assembly)}$$

$$x_2 + d_4^- - d_4^+ = 7 \quad \text{(Ceiling Fan)}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ \quad \text{(Non-negativity)}$$

Harrison Electric Problem using Simplex Method

- Initial table is given as follows:

C_j	SOLUTION MIX	0	0	P_1	P_2	0	P_4	0	0	P_3	0	QUANTITY
		X_1	X_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_2^+	d_3^+	d_4^+	
P_1	d_1^-	7	6	1	0	0	0	–1	0	0	0	30
P_2	d_2^-	2	3	0	1	0	0	0	–1	0	0	12
0	d_3^-	6	5	0	0	1	0	0	0	–1	0	30
P_4	d_4^-	0	1	0	0	0	1	0	0	0	–1	7
Z_j	P_4	0	1	0	0	0	1	0	0	0	–1	7
	P_3	0	0	0	0	0	0	0	0	0	0	0
	P_2	2	3	0	1	0	0	0	–1	0	0	12
	P_1	7	6	1	0	0	0	–1	0	0	0	30
$C_j - Z_j$	P_4	0	–1	0	0	0	0	0	0	0	1	
	P_3	0	0	0	0	0	0	0	0	1	0	
	P_2	–2	–3	0	0	0	0	0	1	0	0	
	P_1	–7	–6	0	0	0	0	1	0	0	0	

Pivot column

Harrison Electric Problem using Simplex Method

Second table is given below:

C_j		0	0	P_1	P_2	0	P_4	0	0	P_3	0	
	SOLUTION MIX	X_1	X_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_2^+	d_3^+	d_4^+	QUANTITY
0	X_1	1	6/7	1/7	0	0	0	–1/7	0	0	0	30/7
P_2	d_2^-	0	9/7	–2/7	1	0	0	2/7	–1	0	0	24/7
0	d_3^-	0	–1/7	–6/7	0	1	0	6/7	0	–1	0	30/7
P_4	d_4^-	0	1	0	0	0	1	0	0	0	–1	7
Z_j	P_4	0	1	0	0	0	1	0	0	0	–1	7
	P_3	0	0	0	0	0	0	0	0	0	0	0
	P_2	0	9/7	–2/7	1	0	0	2/7	–1	0	0	24/7
	P_1	0	0	0	0	0	0	0	0	0	0	0
$C_j - Z_j$	P_4	0	–1	0	0	0	0	0	0	0	1	
	P_3	0	0	0	0	0	0	0	0	1	0	
	P_2	0	–9/7	2/7	0	0	0	–2/7	1	0	0	
	P_1	0	0	1	0	0	0	0	0	0	0	

Pivot column

Harrison Electric Problem using Simplex Method

Final solution:

C_j		0	0	P_1	P_2	0	P_4	0	0	P_3	0	
	SOLUTION MIX	X_1	X_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_2^+	d_3^+	d_4^+	QUANTITY
0	d_2^+	8/5	0	0	–1	3/5	0	0	1	–3/5	0	6
0	X_2	6/5	1	0	0	1/5	0	0	0	–1/5	0	6
0	d_1^+	1/5	0	–1	0	6/5	0	1	0	–6/5	0	6
P_4	d_4^-	–6/5	0	0	0	–1/5	1	0	0	1/5	–1	1
Z_j	P_4	–6/5	0	0	0	–1/5	1	0	0	1/5	–1	1
	P_3	0	0	0	0	0	0	0	0	0	0	0
	P_2	0	0	0	0	0	0	0	0	0	0	0
	P_1	0	0	0	0	0	0	0	0	1	0	0
$C_j - Z_j$	P_4	6/5	0	0	0	1/5	0	0	0	–1/5	1	
	P_3	0	0	0	0	0	0	0	0	1	0	
	P_2	0	0	0	1	0	0	0	0	0	0	
	P_1	0	0	1	0	0	0	0	0	0	0	

Thank You