MMATH18-201: Module Theory

Lecture Notes: Chain Conditions

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In the previous chapter we saw that a submodule of a finitely generated module need not be finitely generated, in these notes we characterize such module. Throughout \( R \) denotes a commutative ring with unity. The result of this section are from Chapter 6 and 7 of [1].

1 Noetherian and Artinian Modules

In this section we introduce the notion of Noetherian and Artinian modules and give some equivalent definitions of the same.

**Definition.** Let \( M \) be an \( R \)-module. A sequence \( \{M_i\} \) of submodules of \( M \) is called

(i) *ascending (or increasing)* if \( M_1 \subseteq M_2 \subseteq \cdots \),

(ii) *descending (or decreasing)* if \( M_1 \supseteq M_2 \supseteq \cdots \) and

(iii) *stationary* if for ascending or descending chain there exists some integer \( n > 0 \) such that \( M_n = M_{n+1} = \cdots \).

**Definition.** An \( R \)-module \( M \) is said to satisfy *ascending chain condition* (or a.c.c. for short) if every ascending chain of submodules of \( M \) is stationary.

**Definition.** An \( R \)-module \( M \) is said to satisfy *maximal condition* if every non-empty subset of submodules of \( M \) has a maximal element\(^1\).

The next result show that for an \( R \)-module \( M \), the above two definition are equivalent.

**Theorem 1.1.** For an \( R \)-module \( M \) the following are equivalent:

(i) \( M \) satisfies a.c.c.

(ii) \( M \) satisfies maximal condition.

**Proof.** (i) \( \implies \) (ii). Let \( \Sigma \) be a non-empty set of submodules of \( M \). Assume to the contrary that \( \Sigma \) has no maximal element. Since \( \Sigma \) is non-empty, there exists a submodule (say) \( M_1 \) in \( \Sigma \). As \( \Sigma \) has no maximal element so we can find some submodule \( M_2 \) in \( \Sigma \) such that \( M_1 \subset M_2 \). Again \( M_2 \) cannot be maximal, so there exists some \( M_3 \) in \( \Sigma \) such that \( M_2 \subset M_3 \). Continuing this way, we obtain a strictly ascending chain of submodules of \( M \),

\[ M_1 \subset M_2 \subset M_3 \subset \cdots, \]

\(^1\)An element \( a \) of a partially ordered set \((X, \leq)\) is called a maximal element of \( X \) if there is no element in \( X \) strictly greater than \( a \), i.e., if for some \( x \in X \) such that \( a \leq x \), then \( a = x \).
which is not stationary. This contradicts (i) hence (ii) holds.

(ii) \implies (i). Let

\[ M_1 \subseteq M_2 \subseteq \cdots, \tag{1} \]

be any ascending chain of submodules of \( M \) and let \( \Sigma = \{ M_i \mid i = 1, 2, \ldots \} \). Then \( \Sigma \neq \emptyset \) so by our hypothesis, \( \Sigma \) has a maximal element (say) \( M_n \) for some positive integer \( n \). Now in view of (1), we have that

\[ M_n \subseteq M_k, \quad \text{for every } k > n. \tag{2} \]

But \( M_n \) is a maximal element of \( \Sigma \), therefore (2) implies that \( M_n = M_k \) for every \( k > n \), i.e., the sequence (1) is stationary which proves (ii).

\[ \square \]

**Definition.** An \( R \)-module \( M \) is called **Noetherian** if it satisfies the ascending chain condition or equivalently it satisfies the maximal condition.

The following result gives another definition of Noetherian modules.

**Theorem 1.2.** An \( R \)-module \( M \) is Noetherian if and only if every submodule of \( M \) is finitely generated.

**Proof.** Suppose first that \( M \) is Noetherian, that is it satisfies a.c.c., and let \( N \) be any submodule of \( M \). If \( N = \{0\} \), then it is finitely generated and we are done. So assume that \( N \) is a non-trivial submodule of \( M \). Then there exist some non-zero element in \( N \) (say) \( x_1 \) and let \( N_1 = \langle x_1 \rangle \) be the submodule of generated by \( x_1 \). If \( N = N_1 \), then \( N \) is cyclic and we are done otherwise \( N_1 \subset N \), i.e., there exists some \( x_2 \) in \( N \setminus N_1 \) and let \( N_2 = \langle x_1, x_2 \rangle \). Again if \( N = N_2 \), then we have that \( N \) is finitely generated otherwise continuing in this manner, we obtain an ascending chain of submodules of \( M \)

\[ N_1 \subset N_2 \subset \cdots. \tag{3} \]

Since \( M \) satisfies a.c.c., therefore the chain (3) is stationary, i.e., there exists some positive integer \( k \) such that \( N_k = N_{k+1} = \cdots \), which in view of the construction of \( N_i \)'s implies that \( N = N_k = \langle x_1, x_2, \ldots, x_k \rangle \) and hence \( N \) is finitely generated.

Conversely assume that every submodule of \( M \) is finitely generated, we need to show that \( M \) satisfies a.c.c. Let

\[ M_1 \subseteq M_2 \subseteq \cdots, \tag{4} \]
be any ascending chain of submodules of $M$ and let $N = \bigcup_{i \in \mathbb{N}} M_i$, verify that $N$ is a submodule of $M$. Then in view of the hypothesis $N$ is finitely generated (say) by $x_1, x_2, \ldots, x_n$, i.e., $\langle x_1, x_2, \ldots, x_n \rangle = N = \bigcup_{i \in \mathbb{N}} M_i$, which implies that there exist positive integers $i_1, i_2, \ldots, i_n$ such that $x_k \in M_{i_k}$ for $1 \leq k \leq n$. Let $m = \max\{i_1, i_2, \ldots, i_n\}$, then in view of (4), we have that $M_{i_k} \subseteq M_m$, i.e., $x_1, x_2, \ldots, x_n \in M_m$ consequently $N = \langle x_1, x_2, \ldots, x_n \rangle \subseteq M_m$. The reverse inclusion $M_m \subseteq N$ follows from the fact that $N = \bigcup_{i \in \mathbb{N}} M_i$, hence we have that $N = M_m$. Now $\bigcup_{i \in \mathbb{N}} M_i = N = M_m$ together with (4) implies that $M_i = M_m$ for every $i \geq m$. Thus the chain (4) is stationary, i.e., $M$ satisfies a.c.c. \qed

Exercise 1.3. Show that the above result can also be proved if instead of a.c.c., we use maximal condition as the definition of a Noetherian module.

We also have some similar result for descending chain of submodules of an $R$-module $M$.

**Definition.** An $R$-module $M$ is said to satisfy descending chain condition (or d.c.c. for short) if every descending chain of submodules of $M$ is stationary.

**Definition.** An $R$-module $M$ is said to satisfy minimal condition if every non-empty subset of submodules of $M$ has a minimal element\(^2\).

The following result shows that the above two definitions are equivalent, whose proof is on the similar lines as that of Theorem 1.1 and left as an exercise.

**Theorem 1.4.** For an $R$-module $M$ the following are equivalent:

(i) $M$ satisfies d.c.c.

(ii) $M$ satisfies minimal condition.

**Definition.** An $R$-module $M$ is called Artinian if it satisfies the descending chain condition or equivalently it satisfies the minimal condition.

The next result give another characterization of Noetherian (respectively Artinian) modules in terms of exact sequences.

\(^2\)An element $a$ of a partially ordered set $(X, \leq)$ is called a minimal element of $X$ if there is no element in $X$ strictly smaller than $a$, i.e., if for some $x \in X$ such that $x \leq a$, then $x = a$. 

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Theorem 1.5. Let

\[ 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0 \]  

be a short exact sequence of \( R \)-modules and \( R \)-homomorphisms. Then \( M \) is Noetherian (respectively Artinian) if and only if both \( M' \) and \( M'' \) are Noetherian (respectively Artinian).

Proof. Suppose first that \( M \) is Noetherian and let

\[ M'_1 \subseteq M'_2 \subseteq \cdots, \]  

be any ascending chain of submodules of \( M' \). Then on applying the \( R \)-homomorphism \( \alpha \) to (6), we obtain an ascending chain of submodules of \( M' \)

\[ (M'_1) \alpha \subseteq (M'_2) \alpha \subseteq \cdots, \]  

because if \( x \in (M'_1) \alpha \), then \( (x)\alpha^{-1} \in M'_1 \subseteq M''_2 \) which implies that \( x = ((x)\alpha^{-1}) \alpha \in (M''_2) \alpha \). Since \( M \) is Noetherian, therefore the chain (7) is stationary, i.e., there exists some positive integer \( n \) such that

\[ (M'_n) \alpha = (M'_{n+1}) \alpha = \cdots. \]  

As the given sequence (5) is exact, so \( \alpha \) is injective and hence an isomorphism when restricted to the image, which implies that \( ((N') \alpha)^{-1} = N' \) for every submodule \( N' \) of \( M' \). Consequently on applying \( \alpha^{-1} \) to (8), we get that

\[ M'_n = M'_{n+1} = \cdots, \]  

i.e., the chain (6) is stationary. Therefore \( M' \) satisfies a.c.c., and hence is a Noetherian module.

Let

\[ M''_1 \subseteq M''_2 \subseteq \cdots, \]  

be any ascending chain of submodules of \( M'' \), then on applying \( \beta^{-1} \) we get

\[ (M''_1) \beta^{-1} \subseteq (M''_2) \beta^{-1} \subseteq \cdots \]  

is an ascending chain of submodules of \( M \) ( \( \because \) if \( x \in (M''_1) \beta^{-1} \), then \( (x)\beta \in M''_1 \subseteq M''_2 \), which implies that \( x \in (M''_2) \beta^{-1} \)). As \( M \) is Noetherian therefore

\[ (M''_m) \beta^{-1} = (M''_{m+1}) \beta^{-1} = \cdots \]  

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for some positive integer \( m \). Since \( \beta \) is surjective (\( \because (5) \) is exact), so we have that \((N''\beta^{-1})\beta = N''\) for every submodule \( N'' \) of \( M'' \) (Why?). Therefore on applying \( \beta \) to (11), we get that

\[
M''_m = M''_{m+1} = \cdots,
\]

i.e., the chain (9) is stationary. Hence \( M'' \) satisfies d.c.c., and is a Noetherian module.

Conversely assume that both \( M' \) and \( M'' \) are Noetherian, we need to show that \( M \) is also Noetherian. Let

\[
M_1 \subseteq M_2 \subseteq \cdots, \tag{12}
\]

be any ascending chain of submodules of \( M \), then on applying \( \alpha^{-1} \) and \( \beta \) to (12), we obtain

\[
(M_1)\alpha^{-1} \subseteq (M_2)\alpha^{-1} \subseteq \cdots,
\]

and

\[
(M_1)\beta \subseteq (M_2)\beta \subseteq \cdots,
\]

ascending chain of submodules of \( M' \) and \( M'' \) respectively. Since both \( M' \) and \( M'' \) are Noetherian the above chains must be stationary, i.e., there exist positive integers \( m \) and \( n \) such that

\[
(M_m)\alpha^{-1} = (M_{m+1})\alpha^{-1} = \cdots \quad \text{and} \quad (M_n)\beta = (M_{n+1})\beta = \cdots.
\]

We can assume without loss of generality that \( n \geq m \) then

\[
(M_n)\alpha^{-1} = (M_{n+1})\alpha^{-1} = \cdots \tag{13}
\]

and

\[
(M_n)\beta = (M_{n+1})\beta = \cdots. \tag{14}
\]

Now in view of (12), \( M_n \subseteq M_k \) for every \( k > n \), so to prove that the ascending chain (12) is stationary it is enough to show that \( M_k \subseteq M_n \) for every \( k > n \).

Let \( k > n \) and \( x \in M_k \). Then \((x)\beta \in (M_k)\beta = (M_n)\beta\) (from (13)), so there exist some \( y \in M_n \) such that \((x)\beta = (y)\beta\), which implies that \((x-y)\beta = 0\), i.e., \( x-y \in \ker \beta = \img \alpha \) (\( \because (5) \) is exact). Consequently there exists some \( z \in M' \) such that \((z)\alpha = x-y\). As \( x \in M_k \) and \( y \in M_n \subseteq M_k \), we have that \((z)\alpha = x-y \in M_k\), which together with (13) implies that \( z \in (M_k)\alpha^{-1} = (M_n)\alpha^{-1} \),
i.e., \((z)\alpha \in M_n\). Now \(y, (z)\alpha \in M_n\) implies that \(x = y + (z)\alpha \in M_n\), hence we have that \(M_k \subseteq M_n\). Thus \(M_k = M_n\) for every \(k > n\), proving that (12) is stationary.

The proof for Artinian case is similar and is left as an exercise. We now give some applications of the above result.

**Corollary 1.6.** If \(N\) is a submodule of an \(R\)-module \(M\), then \(M\) is Noetherian (respectively Artinian) if and only if both \(N\) and \(M/N\) are Noetherian (respectively Artinian).

**Proof.** Since the sequence

\[
0 \rightarrow N \xrightarrow{\alpha} M \xrightarrow{\pi} M/N \rightarrow 0,
\]

where \(\alpha\) is the inclusion map and \(\pi\) the canonical projection, is exact the result follow immediately from the theorem.

**Remark 1.7.** The above result says that submodules and quotient of a Noetherian (respectively Artinian) module are Noetherian (respectively Artinian). In particular, we have that homomorphic image of Noetherian (respectively Artinian) module is Noetherian (respectively Artinian). For if \(f : M \rightarrow M'\) is an \(R\)-homomorphism, where \(M\) is Noetherian, then by First Isomorphism Theorem, \(\text{img } f = (M)f \cong M/\ker f\) is Noetherian.

**Corollary 1.8.** Let \(M_1, M_2, \ldots, M_n\) be submodules of an \(R\)-module \(M\). If each \(M_i\) is Noetherian (respectively Artinian), then so is \(\sum_{i=1}^{n} M_i\).

**Proof.** By induction it is enough to prove the result for \(n = 2\), i.e., if \(M_1\) and \(M_2\) are Noetherian submodules of \(M\), then so is \(M_1 + M_2\). Using Second Isomorphism Theorem, we have that

\[
(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2).
\]

Since \(M_1 \cap M_2\) is a submodule of \(M_2\) and \(M_2\) is Noetherian, therefore by Corollary 1.6, the quotient \(M_2/(M_1 \cap M_2)\) and consequently \((M_1 + M_2)/M_1\) are Noetherian. Again Corollary 1.6 together with the fact that both \(M_1\) and \((M_1 + M_2)/M_1\) are Noetherian now implies that \(M_1 + M_2\) is Noetherian.
Corollary 1.9. If $M_1, M_2, \ldots, M_n$ are Noetherian (respectively Artinian) modules, then so is $\bigoplus_{i=1}^{n} M_i$.

Proof. By induction it is sufficient (Why?) to prove the result for $n = 2$. For the modules $M_1$ and $M_2$, consider the sequence

$$0 \longrightarrow M_1 \xrightarrow{\eta_1} M_1 \oplus M_2 \xrightarrow{\pi_2} M_2 \longrightarrow 0,$$

where $\eta_1$ and $\pi_2$ denote the canonical injection and projection on the first and second coordinates respectively. Verify that the above sequence is indeed exact. The result now follows immediately from Theorem 1.5.

Examples

1. From linear algebra, we know that every subspace of a finite dimensional vector space over a field is finite dimensional. So all finite dimensional vector spaces are Noetherian. In the next section we will see that they are also Artinian (see Proposition 2.5).

2. Let $G$ be a finite group. Then $G$ is a $\mathbb{Z}$-module and all submodules of $G$ are the subgroups. Now $G$ being a finite group has only finitely many subgroups and hence submodules so every ascending and descending chain of submodules must be stationary, i.e., $G$ is both Noetherian and Artinian.

3. The submodules of the $\mathbb{Z}$-module $\mathbb{Z}$ are nothing but the ideals of $\mathbb{Z}$ and $\mathbb{Z}$ is a PID, so it is Noetherian. But $\mathbb{Z}$ is not Artinian because the chain

$$\langle 2 \rangle \supset \langle 2^2 \rangle \supset \cdots \supset \langle 2^n \rangle \supset \cdots,$$

never stops. More generally every infinite cyclic group considered as a $\mathbb{Z}$-module satisfies a.c.c. but not d.c.c.

4. Since every polynomial ring $K[X]$ in an indeterminate $X$ over a field $K$ is a PID (Why?), so the ring $K[X]$ considered as a module over itself is Noetherian. But $K[X]$ is not Artinian, because the descending chain of ideals

$$\langle X \rangle \supset \langle X^2 \rangle \supset \cdots \supset \langle X^n \rangle \supset \cdots,$$

never stops. What can you say about the $K$-module $K[X]$ or the $\mathbb{Z}$-module $\mathbb{Z}[X]$?
5. Let $p$ be a prime number and

$$\mathbb{Q}_p = \left\{ \frac{m}{p^r} \mid r, m \in \mathbb{Z}, r \geq 0 \right\}.$$ 

Then $\mathbb{Q}_p$ is a subgroup of the additive group $\mathbb{Q}$ of rationals containing $\mathbb{Z}$. Consider the quotient group $G = \mathbb{Q}_p/\mathbb{Z}$, we will prove that the group $G$, (as a $\mathbb{Z}$ module) is Artinian but not Noetherian. Let $H$ be any proper non-trivial subgroup of $G$, then by Correspondence Theorem there exist a unique subgroup (say) $K$ of $\mathbb{Q}_p$ containing $\mathbb{Z}$ such that $H = K/\mathbb{Z}$. We first show that $\frac{m}{p^r} \in K$, if and only if $\frac{1}{p^r} \in K$. If $\frac{1}{p^r} \in K$, then $\frac{m}{p^r} \in K$ for every integer $m$ ($\because K$ being a group is closed under additions).

So let $\frac{m}{p^r} \in K \setminus \mathbb{Z}$, then without loss of generality we can assume that $\gcd(m, p^r) = 1$ and $r > 0$. Now $\gcd(m, p^r) = 1$ implies that there exists some integers $a, b$ such that $1 = am + bp^r$, i.e.,

$$\frac{1}{p^r} = a \frac{m}{p^r} + b \in K \quad (\because \mathbb{Z} \subseteq K).$$

Since $K \subset \mathbb{Q}_p$, therefore there is some positive integer $t$ such that $\frac{1}{p^t} \notin K$, and for any $t' > t$

$$\frac{1}{p^t} = \frac{p^{t'-t}}{p^{t'**:t}} \quad \Rightarrow \quad \frac{1}{p^{t'}} \notin K.$$ 

Hence there exists a positive integer (say) $n$, such that

$$\frac{1}{p^n} \in K \quad \text{but} \quad \frac{1}{p^{n+1}} \notin K. \quad (15)$$

From (15), it follows that $\left\langle \frac{1}{p^n} \right\rangle \subseteq K$. In fact $K = \left\langle \frac{1}{p^n} \right\rangle$, because if $\frac{m}{p^r} \in K$, where $p \nmid m$, then $\frac{1}{p^r} \in K$ consequently $r \leq n$ and

$$\frac{m}{p^r} = \frac{mp^{n-r}}{p^n} \in \left\langle \frac{1}{p^n} \right\rangle.$$ 

Therefore

$$H = K/\mathbb{Z} = \left\langle \frac{1}{p^n} + \mathbb{Z} \right\rangle.$$ 

and $o(H) = p^n$ (Why?). As $H$ was an arbitrary subgroup of $G$, therefore, from what we have shown above, it follows that the only proper subgroups
of $G$ are of the form (say) $G_n = \langle \frac{1}{p^n} + \mathbb{Z} \rangle$, and $G$ has exactly one such subgroup of order $p^n$ for each $n \geq 0$. Now the chain

$$G_0 \subset G_1 \subset \cdots \subset G_n \subset \cdots,$$

is a strictly ascending chain of subgroups of $G$, which never stops, so $G$ cannot be Noetherian. But $G$ is Artinian because for any $n \geq 0$, $G_n$ is of order $p^n$ so by Lagrange’s Theorem any descending chain of subgroups of $G_n$ must necessarily stops.

6. The sequence of $\mathbb{Z}$-module

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Q}_p \xrightarrow{\pi} \mathbb{Q}_p/\mathbb{Z} \longrightarrow 0$$

is a short exact sequence, where $\alpha$ is the inclusion map and $\pi$ the canonical projection. Since $\mathbb{Z}$ does not satisfy d.c.c. and a.c.c. fails for $\mathbb{Q}_p/\mathbb{Z}$ therefore by Theorem 1.5, it follows that $\mathbb{Q}_p$ is neither Noetherian nor Artinian.

7. The set of rationals $\mathbb{Q}$ considered as a $\mathbb{Z}$-module is not finitely generated (Why?) so it cannot be Noetherian (see Theorem 1.2). It cannot be Artinian either because it has a submodule which is not Artinian, namely $\mathbb{Z}$.

**Definition.** A commutative ring $R$ is called *Noetherian* (respectively *Artinian*) if it satisfies the ascending chain condition (respectively the descending chain condition) for ideals of $R$.

**Theorem 1.10.** For a commutative ring $R$, the following statements are equivalent.

(i) $R$ is Noetherian, i.e., satisfies a.c.c. for ideals of $R$.

(ii) Every non-empty subset of ideals of $R$ has a maximal ideal, i.e., satisfies maximal condition for ideals of $R$.

(iii) Every ideal of $R$ is finitely generated.

**Proof.** If the ring $R$ is considered as a module over itself, then its submodules are nothing but the ideals of the ring $R$. The result now follows immediately from Theorem 1.2 and Exercise 1.3. \qed
Theorem 1.11. For a commutative ring $R$, the following statements are equivalent.

(i) $R$ is Artinian, i.e., satisfies d.c.c. for ideals of $R$.

(ii) Every non-empty subset of ideals of $R$ has a minimal element, i.e., satisfies minimal condition for ideals of $R$.

Proof. The ring $R$ can be considered as a module over itself and the submodules of the $R$-module $R$ are nothing but the ideals of the ring $R$. The result now follows immediately from Theorem 1.4.

Proposition 1.12. If $R$ is a Noetherian ring and $I$ an ideal of $R$, then the quotient ring $R/I$ is also Noetherian.

Proof. Considering $R$ as a module over itself, we have that $R$ is a Noetherian $R$-module and the ideal $I$ is now a submodule of $R$. Since submodules and quotient of Noetherian modules are Noetherian (see Corollary 1.6), we have that $R/I$ is a Noetherian $R$-module, i.e., every submodule of $R/I$ is finitely generated. But a submodule of $R/I$ is of the form $J/I$ for some ideal $J$ of $R$ containing $I$, hence every ideal of the quotient ring $R/I$ is finitely generated, so $R/I$ is a Noetherian ring.

In fact from the above result we can easily deduce that a homomorphic image of Noetherian ring is again Noetherian. A similar result also holds for Artinian rings.

Theorem 1.13. Let $R$ be a Noetherian (respectively Artinian) ring. If $M$ is a finitely generated $R$-module, then $M$ is Noetherian (respectively Artinian).

Proof. Suppose that $M$ is generated as $R$-module by the elements $x_1, x_2, \ldots, x_n$. Then $M = \sum_{i=1}^{n} \langle x_i \rangle$, where $\langle x_i \rangle = Rx_i$ is the cyclic submodule of $M$ generated by $x_i$. Since a finite sum of Noetherian module is Noetherian (see Corollary 1.8), so to prove that $M$ is Noetherian it is enough to show that each $\langle x_i \rangle$ is Noetherian. Consider the map $f : R \longrightarrow Rx_i = \langle x_i \rangle$ defined by $(r)f = rx_i, r \in R$. Clearly (verify) $f$ is a surjective $R$-module homomorphism, therefore by First Isomorphism Theorem, we have that $R/\ker f \cong Rx_i$. Since $R$ is a Noetherian ring and hence a Noetherian $R$-module, the result now follows from fact that the quotient and homomorphic image of a Noetherian module is again Noetherian.
Remark 1.14. Let $M$ be a Noetherian $R$-module, then every submodule of $M$ and hence $M$ itself must be finitely generated. Therefore over Noetherian rings the converse of the above result also holds, i.e., If $R$ is Noetherian ring, then an $R$-module $M$ is Noetherian if and only if $M$ is finitely generated.

Proposition 1.15. Let $M$ be a finitely generated module over a Noetherian ring $R$. Then $\text{Hom}_R(M, R)$ is a finitely generated $R$-module.

Proof. Suppose $M$ is generated as an $R$-module by the elements $x_1, x_2, \ldots, x_n$. Then the mapping $\beta : R^n \rightarrow M$ defined by

$$(r_1, r_2, \ldots, r_n) \beta = \sum_{i=1}^{n} r_i x_i$$

is a surjective $R$-homomorphism. Therefore, we have a short exact sequence

$$0 \rightarrow \ker \beta \rightarrow R^n \xrightarrow{\beta} M \rightarrow 0,$$

where $\alpha$ is the inclusion map. On applying the contravariant Hom functor $\text{Hom}_R(-, R)$, we obtain a left exact sequence

$$0 \rightarrow \text{Hom}_R(M, R) \rightarrow \text{Hom}_R(R^n, R) \xrightarrow{\alpha^*} \text{Hom}_R(\ker \beta, R) \rightarrow 0,$$

where $(f)\beta^* = \beta f, \; \forall \; f \in \text{Hom}_R(M, R)$. As $\beta^*$ is injective, we have that $\text{Hom}_R(M, R)$ is isomorphic to a submodule of $\text{Hom}_R(R^n, R)$. But

$$\text{Hom}_R(R^n, R) \cong \oplus \text{Hom}_R(R, R) \cong R^n.$$

Since $R$ is Noetherian, so is the direct sum $R^n$. Hence every submodule of $R^n$ is finitely generated. Consequently $\text{Hom}_R(M, R)$ is finitely generated. □

In fact, we have a more general result

Theorem 1.16. If $R$ is a Noetherian ring and $M$ and $N$ are finitely generated $R$-modules, then $\text{Hom}_R(M, N)$ is a finitely generated $R$-module.

Proof. For proof see Theorem 4.3 of [2]. □

Corollary 1.17. If $R$ is a Noetherian ring and $M$ and $N$ are finitely generated $R$-modules, then $\text{Hom}_R(M, N)$ is a Noetherian $R$-module.
Proof. From Theorem 1.16, we have that \( \text{Hom}_R(M, N) \) is a finitely generated \( R \)-module. The result now follows from the converse of Theorem 1.13 given in Remark 1.14.

Another interesting property of Noetherian integral domains is the following

**Theorem 1.18.** Let \( R \) be a Noetherian integral domain. Then every non-zero non-unit element of \( R \) can be factorized into a product of finite number of irreducibles.

*Proof.* For proof see Theorem 3.4 of [2].

**Remark 1.19.** The above result says that factorization into irreducibles is possible in a Noetherian integral domain but the factorization need not be unique, e.g., the integral domain \( \mathbb{Z} [\sqrt{-5}] = \{ a + b\sqrt{-5} \mid a, b \in \mathbb{Z} \} \) is Noetherian but not a unique factorization domain for \( 6 = 2 \cdot 3 = (1 + \sqrt{-5}) (1 + \sqrt{-5}) \).

**Examples**

1. Every field is both a Noetherian and Artinian.

2. For each positive integer \( n \), the ring \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \) of integers modulo \( n \) is both Noetherian and Artinian (Why?).

3. Every PID is a Noetherian ring. However a PID may not be Artinian, e.g., \( \mathbb{Z} \) is a PID, which is not Artinian.

4. Let \( K \) be a field and \( K[X] \) the polynomial ring. Then \( K[X] \) is a PID and hence Noetherian. But \( K[X] \) is not Artinian as the following descending chain of ideals never stops

\[
\langle X \rangle \supset \langle X^2 \rangle \supset \langle X^4 \rangle \supset \cdots.
\]

In fact by Hilbert Basis Theorem (see Theorem 3.1) the polynomial ring \( K[X_1, X_2, \ldots, X_n] \) is also Noetherian, which is again not Artinian as the descending chain of ideals

\[
\langle X_1 \rangle \supset \langle X_1^2 \rangle \supset \langle X_1^4 \rangle \supset \cdots
\]

never stops.
5. The ring $K[X_1, X_2, \ldots]$ of polynomials in infinitely many indeterminates over a field $K$ is neither Noetherian nor Artinian, as the following ascending chains of ideals

$$\langle X_1 \rangle \subset \langle X_1, X_2 \rangle \subset \cdots \subset \langle X_1, X_2, \ldots, X_n \rangle \subset \cdots$$

and descending chains of ideals

$$\langle X_1 \rangle \supset \langle X_1^2 \rangle \supset \langle X_1^3 \rangle \supset \cdots$$

never stops.

2 Composition Series

In this section, we introduce the notion of composition series for modules and give some necessary and sufficient condition for the existence of composition series. As a consequence, we define the length of a module which paves the way for the application of Jordan–Hölder Theorem for modules of finite lengths.

**Definition.** Let $M$ be an $R$-module. A chain in $M$ is a finite sequence $\{M_i\}$ of submodules of $M$ starting with the module $M$ itself and ending with the zero submodule, such that

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = \{0\}.$$  

The number of inclusions or links (‘$\supset$’) is called the **length** of the chain. If $\{N_i\}$ and $\{M_j\}$ are two chains in $M$, then $\{N_i\}$ is called a **refinement** of $\{M_j\}$ if each $M_j$ equals some $N_i$.

**Definition.** A chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = \{0\},$$

of submodules of a module $M$ is called a **composition series** for $M$ if the quotient modules $M_{i-1}/M_i$ ($1 \leq i \leq n$) are simple, i.e., has no submodules other than the trivial one and itself. So the chain is maximal in the sense that we cannot insert any submodule between $M_{i-1}$ and $M_i$, for each $i \in \{1, 2, \ldots, n\}$ or the chain has no proper refinement.
Definition. If a module $M$ has no composition series, then we say that the length of $M$ is infinite. If a module has a composition series, then the least length of a composition series for $M$ is called the length of $M$ and is denoted by $l(M)$.

So for a module $M$ the length $l(M)$ is either infinite or finite depending upon whether $M$ has a composition series or not. If $l(M) < \infty$ then we say that $M$ is of finite length and in this case we prove that $l(M)$ is an invariant for $M$, i.e., every composition series for $M$ has the same length. For this we use the following lemma.

Lemma 2.1. Let $N$ be a proper submodule of a module $M$ of finite length. Then $l(N) < l(M)$.

Proof. Let $l(M) = m$ and

$$M = M_0 \supset M_1 \supset \cdots \supset M_m = \{0\}, \tag{16}$$

a composition series for $M$ of length $m$. For $i \in \{0, 1, \ldots, m\}$ let $N_i = N \cap M_i$. Then from (16), we have a chain

$$N = N_0 \supset N_1 \supset \cdots \supset N_m = \{0\}$$

of submodules of $N$. Consider not the following composition of module homomorphisms

$$N_{i-1} \xrightarrow{\alpha_{i-1}} M_{i-1} \xrightarrow{\pi_{i-1}} M_{i-1}/M_i, \text{ for } i \in \{1, 2, \ldots, m\},$$

where $\alpha_{i-1}$ and $\pi_{i-1}$ denote respectively the inclusion and canonical projections. The kernel of the $\alpha_{i-1} \pi_{i-1}$ is the set of elements of $N_{i-1}$ which lie in $M_i$, i.e.,

$$\ker \alpha_{i-1} \pi_{i-1} = N_{i-1} \cap M_i$$

$$= (N \cap M_{i-1}) \cap M_i$$

$$= N \cap (M_{i-1} \cap M_i)$$

$$= N \cap M_i \quad (\because M_i \subset M_{i-1})$$

$$\therefore \ker \alpha_{i-1} \pi_{i-1} = N_i. \tag{17}$$

So there is an induced injective homomorphism (say) $\phi_{i-1} : N_{i-1}/N_i \rightarrowtail M_{i-1}/M_i$, $(1 \leq i \leq n)$. Since (16) is a composition series, so for each $i$, $M_{i-1}/M_i$ is simple
therefore $\phi_{i-1}(N_{i-1}/N_i)$ is either the trivial module or whole of $M_{i-1}/M_i$, i.e., either $N_{i-1}/N_i$ is trivial or $N_{i-1}/N_i$ is simple (∵ in this case $\phi_{i-1}$ is an isomorphism) which implies that either $N_{i-1} = N_i$ or $N_i$ is maximal in $N_{i-1}$. Hence by deleting repeated terms (if any) in the series

$$N = N_0 \supset N_1 \supset \cdots \supset N_m = \{0\}, \quad (18)$$

we obtain a composition series for $N$ of length at most $m$, i.e., $l(N) \leq l(M)$. If $l(N) = l(M)$, then $N_{i-1} \neq N_i$ $(1 \leq i \leq n)$ therefore for each $i$ we have that $N_{i-1}/N_i$ is simple. Now

$$M_{i-1}/M_i \cong N_{i-1}/N_i \Rightarrow N_{i-1}/(N_{i-1} \cap M_i), \quad \text{(by (17))}$$
$$\cong (N_{i-1} + M_i)/M_i \quad \text{(by Second Isomorphism Theorem)}$$
$$\subseteq M_{i-1}/M_i,$$

which implies that $(N_{i-1} + M_i)/M_i = M_{i-1}/M_i$ consequently $N_{i-1} + M_i = M_{i-1}$ for each $i \in \{1, 2 \ldots, m\}$. Since $M_m = \{0\}$ so for $i = m$ we have

$$N_{m-1} = M_{m-1}.$$

For $i = m - 1,$

$$N_{m-2} + M_{m-1} = N_{m-2} + N_{m-1} = M_{m-2} \quad (\because N_{m-1} = M_{m-1})$$
$$\text{i.e.,} \quad N_{m-2} = M_{m-2} \quad (\because N_{m-1} \subset N_{m-2}).$$

Continuing in this manner we get that $N_{m-3} = M_{m-3}, \ldots, N_1 = M_1$ and finally for $i = 1$, $N_0 = M_0$, i.e., $N = M$, which contradicts the fact that $N$ is a proper submodule of $M$. Hence our assumption that $l(N) = l(M)$ is not possible so $l(N) < l(M)$. \hfill \Box

We now prove that for a module of finite length its length is independent of the choice of composition series.

**Theorem 2.2.** If a module $M$ has a composition series of length $n$, then every composition series of $M$ has length $n$ and every chain can be extended to a composition series.

**Proof.** Since $M$ has a composition series of length $n$, so $l(M) \leq n$. Let

$$M = M_0 \supset M_1 \supset \cdots \supset M_{k-1} \supset M_k = \{0\}, \quad (19)$$

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be any chain in \( M \). Since the inclusions in the above chain are strict so by Lemma 2.1, we have that
\[
l(M) = l(M_0) > l(M_1) > \cdots > l(M_{k-1}) > l(M_k) = 0.
\] (20)

As \( k > k-1 > \cdots > 2 > 1 > 0 \) and \( l(M_i) \) are non-negative integers so from (20) it follows that \( l(M_k) = l(M) \geq k \). Since (19) was an arbitrary chain in \( M \), we have that length of every chain in \( M \) is at most \( l(M) \). But every composition series is also a chain and \( M \) has a composition series of length \( n \) therefore we have that \( n \leq l(M) \), which in view of \( l(M) \leq n \) implies that \( l(M) = n \). Hence every composition series in \( M \) has length \( n \).

Now let
\[
M = N_0 \supset N_1 \supset \cdots \supset N_{k-1} \supset N_k = \{0\},
\] (21)
be any chain in \( M \), then from above we have that \( k \leq n = l(M) \). If \( k = n \), then the chain (21) is a already a composition series because otherwise for some \( i \), \( N_{i-1}/N_i \) is not simple, i.e., we can insert a submodule between \( N_{i-1} \) and \( N_i \), resulting in a chain of length more than \( n \) which is not possible. So assume that \( k < n \). Since all composition series of \( M \) have same length, we have that (21) cannot be a composition series, i.e., for some \( i \in \{1, 2, \ldots, k\} \), \( N_i \) is not a maximal submodule of \( N_{i-1} \). So there exist a submodule (say) \( N_i' \) such that \( N_{i-1} \supset N_i' \supset N_i \) and we obtain a chain of length \( k + 1 \). If \( k + 1 = n \), then this chain becomes a composition series and we are done otherwise we repeat this procedure until we obtain a chain of length \( n \), which must be a composition series.

We now recall the Jordan–Hölder Theorem for finite groups. For proof see Theorem 22, Chapter 3 of [3] or Theorem 5.12 of [4].

**Jordan–Hölder Theorem.** Any two composition series \( \{G_i\}_{0 \leq i \leq n} \) and \( \{G'_i\}_{0 \leq i \leq n} \) of a group \( G \) are equivalent, i.e., there is a one to one correspondence between the set of quotients \( \{G_{i-1}/G_i\}_{0 \leq i \leq n} \) and the set of quotients \( \{G'_{i-1}/G'_i\}_{0 \leq i \leq n} \) such that the corresponding quotients are isomorphic.

In view of Theorem 2.2, it follows that the Jordan–Hölder Theorem is also applicable to modules of finite length. The proof is the same as that for finite groups.

The following result give some necessary and sufficient condition for an \( R \)-module to have a composition series.
Theorem 2.3. An $R$-module $M$ has a composition series if and only if it satisfies both chain conditions (a.c.c., and d.c.c).

Proof. Suppose first that $M$ has a composition series of length $n$. Then in view of Theorem 2.2, $l(M) = n$ and every chain in $M$ has length at most $n$. Therefore we cannot have a strictly ascending or descending chain of submodules of $M$ of length more than $n$. Because if

$$N_1 \subset N_2 \subset \cdots,$$

is a strictly ascending chain of submodules of $M$ consisting of more than $n$ submodules, then on adding the trivial submodule and the module itself, we obtain a chain

$$\{0\} = N_0 \subseteq N_1 \subset N_2 \subset \cdots \subseteq N_{n+1} \subseteq M,$$

of $M$ of length atleast $n + 1$ which contradicts the fact that length of $M$ is $n$. So $M$ satisfies a.c.c. Similarly (verify) we can show that $M$ also satisfies d.c.c.

Conversely assume that $M$ satisfies both a.c.c., and d.c.c., we need to show that $M$ has a composition series. If $M = \{0\}$, then nothing to prove, so suppose that $M \neq \{0\}$ and let $\Sigma_0$ be the set of all proper submodules of $M$. Since $M$ is Noetherian (\because it satisfies a.c.c.), so $\Sigma_0$ has a maximal element (say) $M_1$. By definition of $\Sigma_0$, $M_1$ is a maximal submodule of $M$. Now if $M_1 = \{0\}$, then we have a composition series $M = M_0 \supset M_1 = \{0\}$ of length 1 and we are done. Otherwise let $M_1 \neq \{0\}$, then it is also Noetherian and the set $\Sigma_1$ of all proper submodules of $M_1$ has a maximal element (say) $M_2$, which must be a maximal submodule of $M_1$. Again if $M_2 = \{0\}$, we obtain a composition series

$$M = M_0 \supset M_1 \subset M_2 = \{0\},$$

of length 2. Otherwise continuing in this manner, we get a strictly descending chain

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{i-1} \supset M_i \supset \cdots,$$

of submodules of $M$ such that $M_i$ is a maximal submodule of $M_{i-1}$, for $i = 1, 2, \ldots$. Since $M$ satisfies d.c.c., therefore there exists some positive integer $n$ such that $M_k = M_n$ for every $k > n$, which implies that either $M_n = \{0\}$ or the set $\Sigma_{n+1}$ of all proper submodules of $M_n$ has maximal element the trivial submodule, i.e., we obtain a composition series of length $n$ or $n + 1$. In either case we have that $M$ has a composition series and the result follows. \qed

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From the above theorem, we conclude that a module is of finite length if and only if it satisfies both a.c.c. and d.c.c.

**Proposition 2.4.** The length of a module is an additive function on the class of modules of finite lengths.

**Proof.** To prove the result, we need to show that if $M'$ and $M''$ are two $R$-modules of finite length, then $l(M' \oplus M'') = l(M') + l(M'')$, which will follow (Why?) once we show that if

$$0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0 \quad (22)$$

is a short exact sequence, then

$$l(M) = l(M') + l(M''). \quad (23)$$

Let $l(M') = r$ and $l(M'') = s$, then $M'$ and $M''$ have composition series of lengths $r$ and $s$ respectively (say)

$$M' = M'_0 \supset M'_1 \supset M'_2 \supset \cdots \supset M'_r = \{0\}, \quad (24)$$

and

$$M'' = M''_0 \supset M''_1 \supset M''_2 \supset \cdots \supset M''_s = \{0\}. \quad (25)$$

Since $\alpha$ is injective (Why? (22) is exact), so on applying $\alpha$ to the chain (24) all the inclusions remain strict (Why?), i.e.,

$$\Im \alpha = (M')\alpha = (M_0)\alpha' \supset (M'_1)\alpha \supset (M'_2)\alpha \supset \cdots \supset (M'_r)\alpha = \{0\}. \quad (26)$$

**Claim 1.** For each $i \in \{1, 2, \ldots, r\}$, $(M'_i)\alpha$ is a maximal submodule of $(M'_{i-1})\alpha$.

Suppose if possible that there is some $i \in \{1, 2, \ldots, r\}$ such that

$$(M'_i)\alpha \subset N \subset (M'_{i-1})\alpha,$$

for some submodule $N$ of $M$, then $N \subset \Im \alpha$ and since $\alpha$ is injective, we have that $((N')\alpha)\alpha^{-1} = N'$ for every submodule $N'$ of $M'$, which implies that

$$( (M'_i)\alpha )\alpha^{-1} = M'_i \subset (N)\alpha^{-1} \subset M'_{i-1} = ( (M'_{i-1})\alpha )\alpha^{-1}. \quad (27)$$

But $(N)\alpha^{-1}$ is a submodule of $M'$, so the above (strict) inclusion implies that $M'_i$ is not maximal in $M'_{i-1}$ contradicting the fact that (24) is a composition series of $M'$. Hence the claim follows.
By exactness of (22), we have that \( \beta \) is surjective, so on applying \( \beta^{-1} \) to the composition series (23) all the inclusion in the resulting chain remains strict (Why?), i.e.,

\[
M = (M'') \beta^{-1} = (M_0'') \beta^{-1} \supset (M_1'') \beta^{-1} \supset \cdots \supset (M_s'') \beta^{-1} = (\{0\}) \beta^{-1} = \ker \beta.
\]

(27)

**Claim II:** For \( 1 \leq j \leq s \), \( (M_j'') \beta^{-1} \) is a maximal submodule of \( (M_{j-1}'') \beta^{-1} \).

Assume to the contrary that there is some \( j \in \{ 1, 2, \ldots \} \) such that \( (M_j'') \beta^{-1} \subset N \subset (M_{j-1}'') \beta^{-1} \), for some submodule \( N \) of \( M \), then from (27), we have that \( \ker \beta \subset N \), which in view of Correspondence Theorem\(^3\) implies that

\[
((M_j'') \beta^{-1}) \beta = M_j'' \subset (N) \beta \subset M_{j-1}'' = ((M_{j-1}'') \beta^{-1}) \beta.
\]

But \( (N) \beta \) is a submodule of \( M'' \), so the above (strict) inclusion implies that \( M_j'' \) is not a maximal submodule of \( M_{j-1}'' \) contradicting the fact that (25) is a composition series of \( M'' \) and proving the claim.

On using \( \text{Img} \alpha = \ker \beta \) (\( \because \) (22) is exact), together with the chains (26), (27), and the claims I and II, it follows that the following chain

\[
M = (M'') \beta^{-1} = (M_0'') \beta^{-1} \supset (M_1'') \beta^{-1} \supset \cdots \supset (M_s'') \beta^{-1} = (\{0\}) \beta^{-1} = \ker \beta = \text{Img} \alpha
\]

\[
= (M') \alpha = (M_0') \alpha' \supset (M_1') \alpha \supset (M_2') \alpha \supset \cdots \supset (M_r') \alpha = \{0\}
\]

is a composition series of \( M \) of length \( r + s \) and hence \( l(M) = r + s \).

The next result shows that finite dimensional vector spaces satisfies both a.c.c. and d.c.c and hence have finite length.

**Proposition 2.5.** For a vector space \( V \) over a field \( K \), the following statements are equivalent:

(i) \( V \) is finite dimensional.

(ii) \( V \) is of finite length.

(iii) \( V \) satisfies a.c.c.

(iv) \( V \) satisfies d.c.c.

---

\(^3\)Let \( M \) be an \( R \)-module and \( N \) a submodule of \( M \). Then there exists a one to one correspondence between the submodules of \( M \) which contain \( N \) and the submodules of \( M/N \).
Proof. (i) \implies (ii). Let \( n \) be the dimension of \( V \) and \( \{ e_1, e_2, \ldots, e_n \} \) a basis of \( V \). If \( W_i \) denote the subspace of \( V \) generated by \( \{ e_1, e_2, \ldots, e_i \} \), then we have a chain

\[
 V = W_n \supset W_{n-1} \supset \cdots \supset W_1 \supset W_0 = \{0\}.
\]

It can be easily verified (exercise) that the above chain is a composition series of length \( n \), i.e., \( V \) is of finite length. In fact from above we have that the length of \( V \) is same as its dimension.

(ii) \implies (iii) and (iv) follows immediately from Theorem 2.3.

Finally we prove (simultaneously) that (iii) \implies (i) and (iv) \implies (i). Assume to the contrary that \( V \) is not a finite dimensional vector space over \( K \). Then there is a countably infinite linearly independent subset of \( V \) (say) \( \{ e_1, e_2, \ldots \} \).

For each natural number \( n \) let \( U_n = \langle e_1, e_2, \ldots, e_n \rangle \) and \( W_n = \langle e_n, e_{n+1}, \ldots \rangle \), then \( U_n \) and \( W_n \) are subspaces of \( V \) and we have the following ascending and descending chains

\[
 U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq U_{n+1} \subseteq \cdots, \tag{28}
\]

\[
 W_1 \supseteq W_2 \supseteq \cdots \supseteq W_n \supseteq W_{n+1} \supseteq \cdots. \tag{29}
\]

In fact all the inclusions in the above two chains are strict, because if for some \( n \), \( U_n = U_{n+1} \), then we have that \( e_{n+1} \in U_n = \langle e_1, e_2, \ldots, e_n \rangle \), which implies that \( e_{n+1} \) can be expressed as a \( K \)-linear combination of \( e_1, e_2, \ldots, e_n \) contradicting the fact that \( \{ e_1, e_2, \ldots, e_n, e_{n+1} \} \) is linearly independent. Similarly if \( W_n = W_{n+1} \) for some \( n \), then \( e_n \in W_{n+1} = \langle e_{n+1}, e_{n+2}, \ldots \rangle \), i.e., there exist finitely many non-zero scalars (say) \( a_{j_1}, a_{j_2}, \ldots, a_{j_m} \in K \) such that \( e_n = \sum_{i=1}^{m} a_{j_i} e_{n+j_i} \), contradicting the fact that \( e_n, e_{n+j_1}, \ldots, e_{n+j_m} \) are linearly independent over \( K \). Therefore, we have that (28) and (29) are strictly ascending and descending chain of subspaces of \( V \) which never stops contradicting the hypothesis that \( V \) satisfies a.c.c. and d.c.c. Hence our assumption that \( V \) is not finite dimensional is false and thus (i) follows. \( \square \)

3 Hilbert Basis Theorem

The following result is known as the Hilbert Basis Theorem which says that a polynomial ring over a Noetherian ring is Noetherian.
Theorem 3.1. Let $R$ be a commutative ring with identity. If $R$ is Noetherian, then so is the polynomial ring $R[X]$.

Proof. To prove the result we will show that every ideal of $R[X]$ is finitely generated. Let $I$ be any non-zero ideal of $R[X]$ and let $L$ denote the set of leading coefficients of all the elements in $I$. We first show that $L$ is an ideal of $R$. Since $I$ contains the zero polynomial, so $0 \in L$. For any $r \in R$ and $a, b \in L$, there exist polynomials (say) $f(x) = aX^d + \cdots$, and $g(x) = bX^e + \cdots$ in $I$ with leading coefficients $a$ and $b$ respectively and as $I$ is an ideal of $R[X]$, so the polynomial $rX^e f(x) + X^d g(x) \in I$, and has leading coefficient $ra + b$, therefore we have that $ra + b \in L$, i.e., $L$ is an ideal of $R$. Since $R$ is Noetherian, so the ideal $L$ is finitely generated (say)

$$L = \langle a_1, a_2, \ldots, a_n \rangle. \quad (30)$$

For each $i \in \{1, 2, \ldots, n\}$, let $f_i$ be a polynomial in $I$ of degree $d_i$ with leading coefficient $a_i$ and let $$N = \max\{d_1, d_2, \ldots, d_n\}.$$ Now for each $k \in \{0, 1, \ldots, N - 1\}$ let $L_k$ be the set of all leading coefficients of polynomials in $I$ of degree $k$ together with the zero element. A similar argument as that for $L$ shows that each $L_k$ is also an ideal of $R$, which again must be finitely generated. Let

$$L_k = \langle b_{k,1}, b_{k,2}, \ldots, b_{k,n_k} \rangle, \quad 0 \leq k \leq N - 1 \quad (31)$$

and let $f_{k,i}$ be a polynomial in $I$ of degree $k$ with leading coefficient $b_{k,i}$ for $1 \leq i \leq n_k$. Let

$$I' = \langle \{f_1, f_2, \ldots, f_n\} \cup \{f_{k,i} \mid 0 \leq k \leq N - 1, 1 \leq i \leq n_k\} \rangle. \quad (32)$$

Claim. $I = I'$.

Proof of claim. Since each $f_j, f_{k,i} \in I$, so $I' \subseteq I$. Suppose if possible that $I' \neq I$, then we can choose a non-zero polynomial $f$ in $I$ of minimum degree such that $f \notin I'$. Let $d$ be the degree of the polynomial $f$ and $a$ its leading coefficient.
Then either $d \geq N$ or $d < N$. Suppose first that $d \geq N$. Since $f \in I$, so by definition of $L$, $a \in L$ and therefore in view of (30), we have that

$$a = r_1a_1 + r_2a_2 + \cdots + r_n a_n, \text{ for some } r_1, r_2, \ldots, r_n \in R.$$  

Then the polynomial

$$g(X) = r_1 X^{d-e_1} f_1(X) + r_2 X^{d-e_2} F_2(X) + \cdots + r_n X^{d-e_n} f_n(X)$$

is an element of $I'$ having degree $d$ and leading coefficient $a$, same as that of $f$. As $I' \subset I$, so $f, g \in I$, which implies that $f - g \in I$. But $\deg(f - g) < d$, so by minimality of degree of $f$, we must have that $f - g = 0$, i.e., $f = g \in I'$ which contradicts the choice of the polynomial $f$.

Suppose now that $d < N$. In this case, we have that $a \in L_d$ and therefore in view of (31) there exist some $r_1, r_2, \ldots, r_{n_d} \in R$ such that

$$r_1 b_{d,1} + r_2 b_{d,2} + \cdots + r_{n_d} b_{d,n_d}.$$ 

Then

$$g(X) = r_1 f_{d,1} + r_2 f_{d,2} + \cdots + r_{n_d} f_{d,n_d},$$

is a polynomial in $I'$ with the same degree $d$ and leading coefficient $a$ as that of $f$. Now on arguing as before we must have that $f = g$, which contradicts the choice of $f$. Therefore our assumption that $I' \neq I$ is false. Hence the claim and the result follows.

The following result is an immediate consequence of the Hilbert Basis Theorem and induction.

**Corollary 3.2.** If $R$ is a Noetherian ring, then so is $R[X_1, X_2, \ldots, X_n]$. 

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4 Exercises

1. Let $N_1, N_2$ be submodules of an $R$-module $M$ such that both the quotient modules $M/N_1$ and $M/N_2$ are Noetherian (respectively Artinian). Prove that the quotient module $M/(N_1 \cap N_2)$ is also Noetherian.

2. Let $\alpha : M \rightarrow M$ be a surjective endomorphism of an Noetherian $R$-module $M$. Show that $\alpha$ is an isomorphism.

3. Does the above exercise holds if we interchange surjective by injective.

4. Let $\alpha : M \rightarrow M$ be an injective endomorphism of an Artinian $R$-module $M$. Show that $\alpha$ is an isomorphism.

5. Give an example of a Noetherian ring (respectively Artinian) which has a subring that is not Noetherian (respectively Artinian).

6. Let $R$ be a subring of a ring $S$ such that $R$ is Noetherian and $S$ is a finitely generated $R$-module. Show that $S$ is also Noetherian ring.
References


