# MMATH18-201: Module Theory

## Lecture Notes: Tensor Products

## Contents

In these notes we define tensor products of modules over a commutative ring with unity and prove the universal property. We also discuss various properties of tensor products. Throughout $R$ denotes a commutative ring with unity. The notes are based on Chapter 2 of [1].

# 1 Universal property of Tensor Products

In this section we introduce the notion of tensor product of modules.

**Definition.** Let $M, N$ and $P$ be $R$-modules. A mapping $f : M \times N \longrightarrow P$ is called $R$-bilinear if for each $x \in M$ the map $y \longmapsto (x, y)f$ from $N$ into $P$ is $R$-linear and for each $y \in N$ the map $x \longmapsto (x, y)f$ from $M$ into $P$ is also $R$-linear, i.e.,

$$(a_1x_1 + a_2x_2, y)\, f = a_1\,(x_1, y)\, f + a_2\,(x_2, y)\, f, \quad \forall\ x_1, x_2 \in M \text{ and } \forall\ a_1, a_2 \in R$$

and

$$(x, a_1y_1 + a_2y_2)\, f = a_1\,(x, y_1)\, f + a_2\,(x, y_2)\, f, \quad \forall\ y_1, y_2 \in N \text{ and } \forall\ a_1, a_2 \in R.$$

The motivation[1] for tensor product of two $R$-modules $M$ and $N$ is to construct an $R$-module $T$ with the property that the $R$-bilinear mappings from $M \times N$ into $P$ are in a natural one to one correspondence with the $R$-linear mappings from $T$ into $P$, for all $R$-module $P$, more precisely we have

**Theorem 1.1.** *Let $M$ and $N$ be $R$-modules. Then there exist a pair $(T, g)$ consisting of an $R$-module $T$ and an $R$-bilinear map $g : M \times N \longrightarrow T$ such that for given any $R$-module $P$ and any $R$-bilinear map $f : M \times N \longrightarrow P$, there exist a unique $R$-module homomorphism $f' : T \longrightarrow P$ such that $f = gf'$, i.e., the following diagram commutes*

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, "f'"] \\ & P \end{tikzcd}.$$

*Moreover if $(T', g')$ is another pair with this property, then there exist a unique isomorphism $h : T \longrightarrow T'$ such that $gh = g'$.*

[1]For a different and a more general approach see Chapter 10, Section 4 of [2]. For tensor product of vector spaces see [3].

*Proof. Existence.* Let $F$ denote the free $R$-module with basis the set $M \times N$. Then the elements of $F$ are formal $R$-linear combinations of finitely many elements of $M \times N$, i.e., an arbitrary element of $F$ is of the form

$$\sum_{i=1}^{n} a_i\,(x_i, y_i)\,, \quad a_i \in R \text{ and } (x_i, y_i) \in M \times N.$$

Let $D$ be the submodule of $F$ generated by all elements (formal sums) of the form

$$(x + x', y) - (x, y) - (x', y)\,, \quad (x, y + y') - (x, y) - (x, y')\,,$$

$$(ax, y) - a(x.y) \text{ and } (x, ay) - a(x, y) \quad \text{for } x, x' \in M, y, y' \in N \text{ and } a \in R.$$

Let $T$ denote the quotient module $F/D$ and $x \otimes y$ denotes the equivalence class (coset) to which the basis element $(x, y)$ of $F$ belongs, i.e.,

$$x \otimes y = (x, y) + D.$$

Then $T$ is generated by such elements, so an arbitrary element of $T$ is of the form

$$\sum_{i=1}^{n} a_i\,(x_i \otimes y_i) = \sum_{i=1}^{n} a_i\,((x_i, y_i) + D) = \sum_{i=1}^{n} a_i\,(x_i, y_i) + D,\ (x_i, y_i) \in M \times N, a_i \in R.$$

For any $x, x' \in M$ and $y \in N$, we have

$$\begin{array}{rrcl} & (x + x', y) - (x, y) - (x'y) & \in & D \\ \Longrightarrow & \big((x + x', y) - (x, y) - (x'y)\big) + D & = & D \\ \Longrightarrow & (x + x', y) + D - (x, y) + D - (x'y) + D & = & D \\ \Longrightarrow & (x + x', y) + D & = & (x, y) + D + (x'y) + D, \end{array}$$

i.e.,

$$(x + x') \otimes y = x \otimes y + x' \otimes y. \tag{1}$$

A similar argument shows that for any $x \in M$ and $y, y' \in N$

$$x \otimes (y + y') = x \otimes y + x \otimes y'. \tag{2}$$

Now for any $a \in R$ and $(x, y) \in M \times N$, we have

$$\begin{array}{rrcl} & (ax, y) - a(x, y) & \in & D \\ \Longrightarrow & \big((ax, y) - a(x, y)\big) + D & = & D \\ \Longrightarrow & (ax, y) + D - a(x, y) + D & = & D \\ \Longrightarrow & (ax, y) + D & = & a(x, y) + D, \end{array}$$

i.e.,
$$(ax) \otimes y = a\,(x \otimes y)\,, \tag{3}$$
and similarly
$$x \otimes (ay) = a\,(x \otimes y)\,. \tag{4}$$
Let $g : M \times N \longrightarrow T$ be a map defined by $(x, y)g = x \otimes y$, then from the equations (1), (2), (3) and (4), we have that $g$ is $R$-bilinear. Let $P$ be any $R$-module and $f : M \times N \longrightarrow P$ be any $R$-bilinear map. Since $F$ is a free module with basis $M \times N$, so by the universal property of free modules the map $f$ extends linearly to an $R$-module homomorphism (say) $\bar{f} : F \longrightarrow P$ such that the following diagram commutes
$$\begin{tikzcd} M \times N \arrow[r, "\alpha"] \arrow[dr, "f"'] & F \arrow[d, "\bar{f}"] \\ & P \end{tikzcd}$$
where $\alpha$ is the inclusion map. Define a map $f' : T = F/D \longrightarrow P$ by
$$(x \otimes y)\,f' = (x, y)\bar{f} = (x, y)f \quad (\because\ f, \bar{f} \text{ coincide on } M \times N). \tag{5}$$
To show that $f'$ is well-defined we first prove that $\bar{f}$ maps generators of $D$ to the zero element of $P$. For any $x, x' \in M$ and $y, y' \in N$, we have
$$\begin{aligned} ((x + x', y) - (x, y) - (x', y))\,\bar{f} &= (x + x', y)\bar{f} - (x, y)\bar{f} - (x', y)\bar{f} \quad (\because\ \bar{f} \text{ is } R\text{-linear}) \\ &= (x + x', y)f - (x, y)f - (x', y)f \quad (\text{by } (5)) \\ &= (x, y)f + (x', y)f - (x, y)f - (x, y)f \ (\because f \text{ is } R\text{-bilinear}), \end{aligned}$$
$$\text{i.e.,} \qquad ((x + x', y) - (x, y) - (x', y))\,\bar{f} = 0,$$
and similarly $((x, y + y') - (x, y) - (x, y'))\,\bar{f} = 0$. Also for any $a \in R$
$$\begin{aligned} ((ax, y) - a(x, y))\,\bar{f} &= (ax, y)\bar{f} - a\left((x, y)\bar{f}\right) \quad \because\ \bar{f} \text{ is } R\text{-linear} \\ &= (ax, y)f - a\,((x, y)f) \quad \text{by } (5) \\ &= a\,((x, y)f) - a\,((x, y)f) \quad \because f \text{ is } R\text{-bilinear}, \\ &= 0, \end{aligned}$$
$$\text{i.e.,} \qquad ((ax, y) - a(x, y))\,\bar{f} = 0,$$
and similarly $((x, ay) - a(x, y))\,\bar{f} = 0$. Let $x \otimes y = x' \otimes y'$, then
$$(x, y) + D = (x', y') + D \implies (x, y) - (x', y') \in D,$$

i.e., $(x, y) - (x', y') = \sum_i a_i d_i$, for some generators $d_i$ of $D$ and $a_i \in R$. Now as $\bar{f}$ vanishes on generators of $D$, therefore we have that

$$((x, y) - (x', y'))\, \bar{f} = \left(\sum_i a_i d_i\right) \bar{f} = \sum_i a_i\, (d_i)\, \bar{f} = 0,$$

which implies that $(x, y)\bar{f} = (x', y')\bar{f}$, and hence in view of (5), we have that

$$(x \otimes y) f' = (x' \otimes y') f',$$

i.e., $f'$ is well-defined. The map $f'$ is infact an $R$-module homomorphism (verify). By definition of $g$ and $f'$, we have that for any $(x, y) \in M \times N$

$$(x, y) f = (x \otimes y) f' = ((x, y) g)\, f',$$

i.e., $f = gf'$ or the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, "f'"] \\ & P \end{tikzcd} \tag{6}$$

*Uniqueness of $f'$.* Let $f_1$ be another $R$-module homomorphism from $T$ into $P$ such that $f = gf_1$. Then for any $(x, y) \in M \times N$, we have that

$$(x, y) f = (x, y) g f_1 = ((x, y) g)\, f_1 = (x \otimes y) f_1.$$

Now $f = gf'$ implies that

$$(x, y) f = (x, y) g f' = ((x, y) g)\, f' = (x \otimes y) f'.$$

Therefore from the above two equation, we have that

$$(x \otimes y) f_1 = (x \otimes y) f',$$

which in view of the fact that the elements $x \otimes y$ generates the $R$-module $T$ implies that $f_1 = f'$, i.e., $f'$ is unique.

*Uniqueness of the pair $(T, g)$.* Suppose that $(T', g')$ is another pair with the given property. Taking $(T, g)$ as the initial pair and replacing $(P, f)$ by $(T', g')$

in the commutative diagram (6), we have that there exist a unique $R$-module homomorphism $t : T \longrightarrow T'$ such that

$$gt = g', \qquad \begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "g'"'] & T \arrow[d, "t"] \\ & T' \end{tikzcd} \tag{7}$$

Now taking $(T', g')$ as the initial pair and replacing $(P, f)$ by the pair $(T, g)$ in the commutative diagram (6), we have that there exist a unique $R$-module homomorphism $t' : T' \longrightarrow T$ such that

$$g't' = g, \qquad \begin{tikzcd} M \times N \arrow[r, "g'"] \arrow[dr, "g"'] & T' \arrow[d, "t'"] \\ & T \end{tikzcd} \tag{8}$$

On combining the commutative diagrams (7) and (8), we get

$$g't't = g' \quad \text{and} \quad gtt' = g',$$

i.e., $tt'$ and $t't$ makes the following two diagrams commutative

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "g'"] \arrow[ddr, "g"'] & T \arrow[d, "t"] \arrow[dd, bend left, "1_T"] \\ & T' \arrow[d, "t'"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} M \times N \arrow[r, "g'"] \arrow[dr, "g"] \arrow[ddr, "g'"'] & T' \arrow[d, "t'"] \arrow[dd, bend left, "1_{T'}"] \\ & T \arrow[d, "t"] \\ & T' \end{tikzcd}$$

But $g1_T = g$ and $g'1_{T'} = g'$, i.e., the identity maps $1_T$ and $1_{T'}$ also makes the above two diagram commutative, therefore by the uniqueness property of the pairs $(T, g)$ and $(T', g')$, we must have that

$$tt' = 1_T \quad \text{and} \quad t't = 1_{T'}$$

which implies that $t$ and $t'$ are inverse of each other, i.e., $t$ is a unique isomorphism from $T$ into $T'$. $\square$

**Definition.** Let $M$ and $N$ be $R$-module. The $R$-module $T$ constructed in the above theorem is called the *tensor product* of $M$ and $N$ and is denoted by $M \otimes_R N$ or just by $M \otimes N$ when the underlying ring is evident from the context.

**Remark 1.2.** **1.** From the existence part of the above theorem it immediately follows that for any $x, x' \in M$, $y, y' \in N$ and $a \in R$ the following holds:

(i) $(x + x') \otimes y = x \otimes y + x' \otimes y$,

(ii) $x \otimes (y + y') = x \otimes y + x \otimes y'$,

(iii) $(ax) \otimes y = a\,(x \otimes y) = x \otimes (ay)$.

In particular from (iii), we have that $x \otimes 0 = 0 \otimes y = 0$ (Why?).

**2.** From Theorem 1.1 it is clear that the tensor product $M \otimes N$ is generated as an $R$-module by the "products" $x \otimes y$, where $x \in M$ and $y \in N$. If $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ are family of generators of $M$ and $N$ respectively, then the elements $x_i \otimes y_j$ generates $M \otimes N$. In particular if $M$ and $N$ are finitely generated, then so is $M \otimes N$. However the notation $x \otimes y$ has no meaning unless, we specify the tensor product to which it belongs. For instance the element $2 \otimes \bar{1}$ is a zero in the $\mathbb{Z}$-module $\mathbb{Z} \otimes \mathbb{Z}_2$ ($\because 2 \otimes \bar{1} = 2 \cdot 1 \otimes \bar{1} = 2(1 \otimes \bar{1}) = 1 \otimes 2 \cdot \bar{1} = 1 \otimes \bar{0} = 0$). But is a non-zero in the $\mathbb{Z}$-module $2\mathbb{Z} \otimes \mathbb{Z}_2$. For if $2 \otimes \bar{1} = 0$, then as $2 \otimes \bar{1}$ generates $2\mathbb{Z} \otimes \mathbb{Z}_2$, so it must be the zero module which contradicts Theorem 1.1 (the universal property of tensor products).

**3.** Like bilinear maps we can define trilinear or more generally multi-linear maps and obtain an analogue of Theorem 1.1 for tensor product of three of more modules. For given $R$-modules (say) $M_1, M_2, \ldots, M_n$, for some positive integer $n$, try to restate Theorem 1.1.

# 2 Properties of Tensor Products

In this section, we discuss some properties of the tensor products of modules.

**Property 1.** Tensor product is commutative, i.e., if $M$ and $N$ are two $R$-modules then

$$M \otimes N \cong N \otimes M.$$

*Proof.* For the tensor products $M \otimes N$ and $N \otimes M$, in view of Theorem 1.1, there exist $R$-bilinear maps (say) $g : M \times N \longrightarrow M \otimes N$ and $g' : N \times M \longrightarrow N \otimes M$ such $(x, y)g = x \otimes y$ and $(y, x)g' = y \otimes x$ for some $x \in M$ and $y \in N$. We now define a map $f : M \times N \longrightarrow N \otimes M$ by

$$(x, y)f = y \otimes x \quad (x, y) \in M \times N.$$

Then $f$ is $R$-bilinear (see part 1 of Remark 1.2). Therefore by the universal property of tensor product $M \otimes N$, there exist a unique $R$-module homomorphism (say) $f' : M \otimes N \longrightarrow N \otimes M$ such that $f = gf'$, i.e., the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "f"'] & M \otimes N \arrow[d, "f'"] \\ & N \otimes M \end{tikzcd} .$$

So for any $(x, y) \in M \times N$, we have that $(x, y)f = (x, y)gf' = ((x, y)g)\, f' = (x \otimes y)f'$, which in view of the definition of $f$ implies that

$$(x \otimes y)f' = y \otimes x. \tag{9}$$

Similarly if we define a map $h : N \times M \longrightarrow M \otimes N$ by

$$(y, x)h = x \otimes y \quad (y, x) \in N \times M,$$

then $h$ is $R$-bilinear and by the universal property of the tensor product $N \otimes M$, there exist a unique $R$-module homomorphism (say) $h' : N \otimes M \longrightarrow M \otimes N$ such that $h = g'h'$, i.e., the following diagram commutes

$$\begin{tikzcd} N \times M \arrow[r, "g'"] \arrow[dr, "h"'] & N \otimes M \arrow[d, "h'"] \\ & M \otimes N \end{tikzcd} ,$$

and for any $(y, x) \in N \times M$, we have that

$$(y \otimes x)h' = x \otimes y. \tag{10}$$

Now for any $x \in M$ and $y \in N$ on using (9) and (10), we have that

$$(x \otimes y)\, f'h' = ((x \otimes y)f')\, h' = (y \otimes x)h' = (x \otimes x) = (x \otimes y)1_{M \otimes N}$$

as $x \otimes y$ generates the tensor product $M \otimes N$ the above equation implies that

$$f'h' = 1_{M \otimes N}. \tag{11}$$

Similarly we have that

$$h'f' = 1_{N \otimes M}. \tag{12}$$

Thus from (11) and (12) it now follows that $M \otimes N$ and $M \otimes N$ are isomorphic $R$-modules. $\square$

**Property 2.** Tensor product is associative, i.e., if $M, N$ and $P$ are $R$-modules then

$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P)\,.$$

*Proof.* For each arbitrary but fixed $p \in P$, define $f_p : M \times N \longrightarrow M \otimes (N \otimes P)$ by

$$(x, y) f_p = x \otimes (y \otimes p)\,.$$

Then (verify) $f_p$ is $R$-bilinear. Therefore by the universal property of tensor product $M \otimes N$, there exist a unique $R$-module homomorphism $f'_p : M \otimes N \longrightarrow M \otimes (N \otimes P)$ such that $f_p = g f'_p$, i.e., the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "f_p"'] & M \otimes N \arrow[d, "f'_p"] \\ & M \otimes (N \otimes P) \end{tikzcd} ,$$

where $g : M \times N \longrightarrow M \otimes N$ is $R$-bilinear mapping $(x, y) \longmapsto x \otimes y$. Now for any $(x, y) \in M \times N$, we have that $(x, y) f_p = (x, y) g f'_p = (x \otimes y) f'_p$, which in view of the definition of $f_p$ implies that

$$(x \otimes y) f'_p = x \otimes (y \otimes p)\,. \tag{13}$$

As $p \in P$ was arbitrary, so if $p' \in P$ and $a \in R$, then $p + p', ap \in P$ and by what we have shown above, there exist $R$-module homomorphisms $f'_{p+p}$ and $f'_{ap}$ such that for any $(x, y) \in M \times N$, we have that

$$(x, y) f'_{p+p'} = x \otimes (y \otimes (p + p')) = x \otimes (y \otimes p + y \otimes p') = x \otimes (y \otimes p) + x \otimes (y \otimes p')\,,$$

i.e.,

$$(x, y) f'_{p+p'} = (x, y) f'_p + (x, y) f'_{p'}$$

and $(x, y) f'_{ap} = x \otimes (y \otimes ap) = x \otimes (a(y \otimes p)) = a\,(x \otimes (y \otimes p))\,,$ i.e.,

$$(x, y) f'_{ap} = a(x, y) f'_p.$$

Therefore the function $\phi : (M \otimes N) \times P \longrightarrow M \otimes (N \otimes P)$ given by

$$((x \otimes y), p)\, \phi = (x, y) f'_p$$

is $R$-bilinear. Hence by the universal property of the tensor product $(M \otimes N) \otimes P$, $\phi$ induces a unique $R$-module homomorphism $\phi' : (M \otimes N) \otimes P \longrightarrow M \otimes (N \otimes P)$, such that

$$((x \otimes y) \otimes p)\, \phi' = x \otimes (y \otimes p)\,. \tag{14}$$

Similarly we obtain a unique $R$-module homomorphism $\psi' : M \otimes (N \otimes P) \longrightarrow (M \otimes N) \otimes P$ such that

$$(x \otimes (y \otimes p))\, \psi' = (x \otimes y) \otimes p. \tag{15}$$

Now from (14) and (15) it immediately follows that $\phi'\psi'$ and $1_{((M\otimes N)\otimes P}$ agree on the generators of $(M \otimes N) \otimes P$, i.e., $\phi'\psi' = 1_{(M\otimes N)\otimes P}$. Similarly we have that $\psi'\phi' = 1_{M\otimes(N\otimes P)}$. Thus $\phi'$ and $\psi'$ are inverse of each others, i.e., $\phi'$ is an isomorphism. □

**Proposition 2.1.** *Let $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ be $R$-module homomorphism. Then there is a unique $R$-module homomorphism $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ such that $(x \otimes y)f \otimes g = (x)f \otimes (y)g$ for $(x, y) \in M \times N$.*

*Proof.* Define a map $\phi : M \times N \longrightarrow M' \otimes N'$ by $(x, y)\phi = (x)f \otimes (y)g$. Then (verify) $\phi$ is $R$-bilinear. Therefore by the universal property of the tensor product $M \otimes N$, there exist a unique $R$-module homomorphism $\phi' = f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ such that $\phi = \psi\phi'$, i.e., the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "\psi"] \arrow[dr, "\phi"'] & M \otimes N \arrow[d, "\phi'=f\otimes g"] \\ & M' \otimes N' \end{tikzcd},$$

where $\psi$ is maps $(x, y)$ to $x \otimes y$. Hence for any $(x, y) \in M \times N$, we have that

$$(x, y)\phi = (x, y)\psi\phi' = (x \otimes y)\phi' = (x \otimes y)f \otimes g,$$

which in view of the definition of $\phi$ implies that

$$(x \otimes y)f \otimes g = (x)f \otimes (y)g$$

and the result follows. □

**Corollary 2.2.** *Let $f : M \longrightarrow M'$, $f' : M' \longrightarrow M''$, $g : N \longrightarrow N'$ and $g : N' \longrightarrow N''$ be $R$-module homomorphisms. Then $(f \otimes g)\,(f' \otimes g') = ff' \otimes gg'$.*

*Proof.* From Proposition 2.1, we have that $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ and $f' \otimes g' : M' \otimes N' \longrightarrow M'' \otimes N''$ are $R$-module homomorphism and therefore so is the composition

$$(f \otimes g)\,(f' \otimes g') : M \otimes N \longrightarrow M'' \otimes N''. \tag{16}$$

Again on applying Proposition 2.1 to the compositions $ff' : M \longrightarrow M''$ and $gg' : N \longrightarrow N''$, we have an $R$-module homomorphism

$$ff' \otimes gg' : M \otimes N \longrightarrow M'' \otimes N''. \tag{17}$$

We need to show that the mappings in (16) and (17) are the same. For this it is enough to show that both the mappings coincide on the generators of $M \otimes N$. For any $(x, y) \in M \times N$, $x \otimes y \in M \otimes N$ on applying Proposition 2.1 to (16), we have that

$$(x \otimes y)\,(f \otimes g)\,(f' \otimes g') = ((x)f \otimes (y)g)\,(f' \otimes g') = ((x)f)\,f' \otimes ((y)g)\,g'. \tag{18}$$

Now on applying Proposition 2.1 to (17), we get that

$$(x \otimes y)\,(ff' \otimes gg') = (x)ff' \otimes (y)gg' = ((x)f)\,f' \otimes ((y)g)\,g'. \tag{19}$$

The result now follows form (18) and (19). □

**Corollary 2.3.** *If $1_M : M \longrightarrow M$ and $1_N : N \longrightarrow N$ are identity homomorphisms, then $1_M \otimes 1_N : M \otimes N \longrightarrow M \otimes N$ is the identity homomorphism on $M \otimes N$, i.e., $1_M \otimes 1_N = 1_{M \otimes N}$.*

*Proof.* Exercise. □

**Corollary 2.4.** *If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ be $R$-module isomorphisms, then so is $f \otimes g$ from $M \otimes N$ into $M' \otimes N'$.*

*Proof.* Since $f$ and $g$ are isomorphisms, so let $f' : M' \longrightarrow M$ and $g' : N' \longrightarrow N$ be there inverses. Then $ff' = 1_M, f'f = 1_{M'}$ and $gg' = 1_N, g'g = 1_{N'}$ and hence from Corollary 2.2 and 2.3, we have that

$$(f \otimes g)\,(f' \otimes g') = ff' \otimes gg' = 1_M \otimes 1_N = 1_{M \otimes N}$$

and

$$(f' \otimes g')\,(f \otimes g) = f'f \otimes g'g = 1_{M'} \otimes 1_{N'} = 1_{M' \otimes N'}.$$

From the above two equations it now follows that $f \otimes g$ and $f' \otimes g'$ are inverses of each others, i.e., $f \otimes g$ is an isomorphism. □

**Property 3.** Tensor product distributes over direct sum, i.e., if $M, N$ and $P$ are $R$-modules, then

$$(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)\,.$$

*Proof.* Let $\eta_M : M \longrightarrow M \oplus N$, $\eta_N : N \longrightarrow M \oplus N$ be canonical injections and $\pi_M : M \oplus N \longrightarrow M$, $\pi_N : M \oplus N \longrightarrow N$ be canonical projections. Then the compositions satisfies

$$\eta_M \pi_M = 1_M, \quad \eta_N \pi_N = 1_N, \quad \eta_N \pi_M = 0 \quad \text{and} \quad \eta_M \pi_N = 0.$$

Now on using Corollary 2.2, we have the following compositions

$$M \otimes P \xrightarrow{\eta_M \otimes 1_P} (M \oplus N) \otimes P \xrightarrow{\pi_M \otimes 1_P} M \otimes P$$

$$(\eta_M \otimes 1_P)(\pi_M \otimes 1_P) = \eta_M \pi_M \otimes 1_P 1_P = 1_M \otimes 1_P = 1_{M \otimes P}; \tag{20}$$

$$N \otimes P \xrightarrow{\eta_N \otimes 1_P} (M \oplus N) \otimes P \xrightarrow{\pi_N \otimes 1_P} N \otimes P$$

$$(\eta_N \otimes 1_P)(\pi_N \otimes 1_P) = \eta_N \pi_N \otimes 1_P 1_P = 1_N \otimes 1_P = 1_{N \otimes P}; \tag{21}$$

$$M \otimes P \xrightarrow{\eta_M \otimes 1_P} (M \oplus N) \otimes P \xrightarrow{\pi_N \otimes 1_P} N \otimes P$$

$$(\eta_M \otimes 1_P)(\pi_N \otimes 1_P) = \eta_M \pi_N \otimes 1_P 1_P = 0 \otimes 1_P = 0; \tag{22}$$

$$N \otimes P \xrightarrow{\eta_N \otimes 1_P} (M \oplus N) \otimes P \xrightarrow{\pi_M \otimes 1_P} M \otimes P$$

$$(\eta_N \otimes 1_P)(\pi_M \otimes 1_P) = \eta_N \pi_M \otimes 1_P 1_P = 0 \otimes 1_P = 0. \tag{23}$$

Define a map $f : (M \otimes P) \oplus (N \otimes P) \longrightarrow (M \oplus N) \otimes P$ by

$$(u, v)f = (u)(\eta_M \otimes 1_P) + (v)(\eta_N \otimes 1_P)\,, \quad u \in M \otimes P,\ v \in N \otimes P.$$

Then (verify) $f$ is an $R$-module homomorphism. We now show that $f$ is an isomorphism.

*$f$ is injective.* For any $u \in M \otimes P$ and $v \in N \otimes P$, we have that

$$\begin{aligned}
(u, v)f(\pi_M \otimes 1_P) &= ((u)(\eta_M \otimes 1_P) + (v)(\eta_N \otimes 1_P))(\pi_M \otimes 1_P) \\
&= (u)(\eta_M \otimes 1_P)(\pi_M \otimes 1_P) + (v)(\eta_N \otimes 1_P)(\pi_M \otimes 1_P) \\
&= (u)1_{M \otimes P} + (v)0 && \text{(see (20), (23))}
\end{aligned}$$

i.e.,

$$(u,v)f\,(\pi_M \otimes 1_P) = u. \tag{24}$$

Similarly we have that

$$\begin{aligned}
(u,v)f\,(\pi_N \otimes 1_P) &= ((u)\,(\eta_M \otimes 1_P) + (v)\,(\eta_N \otimes 1_P))\,(\pi_N \otimes 1_P)\\
&= (u)\,(\eta_M \otimes 1_P)\,(\pi_N \otimes 1_P) + (v)\,(\eta_N \otimes 1_P)\,(\pi_N \otimes 1_P)\\
&= (u)0 + (v)1_{N\otimes P} && \text{(see (21), (22))}
\end{aligned}$$

i.e.,

$$(u,v)f\,(\pi_N \otimes 1_P) = v. \tag{25}$$

Now if $(u,v) \in \ker f$, then $(u,v)f = 0$, which in view of (24) and (25), implies that

$$u = (u,v)f\,(\pi_M \otimes 1_P) = (0)\,(\pi_M \otimes 1_P) = 0,$$

and

$$v = (u,v)f\,(\pi_N \otimes 1_P) = (0)\,(\pi_N \otimes 1_P) = 0.$$

Hence $(u.,v) = 0$, i.e., $f$ is one-one.

*$f$ is surjective.* Let $x \in M, y \in N$ and $p \in P$ be arbitrary, then $(x,y)\otimes p \in (M \oplus N) \otimes P$, $x \otimes p \in M \otimes P$ and $y \otimes p \in N \otimes P$. Therefore by the definition of $f$, we have that

$$\begin{aligned}
(x\otimes p, y\otimes p)\,f &= (x\otimes p)\,(\eta_M \otimes 1_P) + (y\otimes p)\,(\eta_N \otimes 1_P)\\
&= (x)\eta_M \otimes (p)1_P + (y)\eta_N \otimes (p)1_P\\
&= (x,0)\otimes p + (0,y)\otimes p\\
\therefore \quad (x\otimes p, y\otimes p)\,f &= (x,y)\otimes p.
\end{aligned}$$

Since the elements $(x,y)\otimes p$ generates $(M\oplus N)\otimes P$, the above equation implies that $f$ is surjective. □

**Property 4.** Let $M$ be an $R$-module. Then on regarding $R$ as a module over itself, we have

$$R \otimes M \cong M.$$

*Proof.* Define a map $f : R \times M \longrightarrow M$ by

$$(a,x)f = ax, \quad a \in R,\ x \in M.$$

By properties of an $R$-module it can be easily verified (exercise) that $f$ is $R$-bilinear. Then by the universal property of the tensor product $R \otimes M$, there exist a unique $R$-module homomorphism $f' : R \otimes M \longrightarrow M$ such that $f = gf'$, i.e., the following diagram commutes

$$\begin{tikzcd} R \times M \arrow[r, "g"] \arrow[dr, "f"'] & R \otimes M \arrow[d, "f'"] \\ & M \end{tikzcd},$$

where $g$ maps $(a, x) \mapsto a \otimes x$. Now for any $(a, x) \in R \times M$, we have that

$$(a, x)f = (a, x)gf' = (a \otimes x)f',$$

which in view of the definition of $f$ implies that

$$(a \otimes x)f' = ax.$$

We now claim that $f'$ is an isomorphism.
*$f$ is surjective.* For any $x \in M$, as $R$ contains identity element, we have that $1 \otimes x \in R \otimes M$ and then

$$(1 \otimes x)f' = 1m = m,$$

so $f'$ is surjective.
*$f$ is injective.* An arbitrary element of $R \otimes M$ is a finite sum of the form

$$\sum_i a_i \otimes x_i = \sum_i a_i (1 \otimes x_i) = \sum_i 1 \otimes (a_i x_i) = 1 \otimes \sum_i a_i x_i,$$

for some $a_i \in R$ and $x_i \in M$. Since $M$ is an $R$-module, so the finite sum $\sum_i a_i x_i = m$ for some $m \in M$ and therefore we have that an arbitrary element of $R \otimes M$ is of the form $1 \otimes m$. Now if $1 \otimes m \in \ker f$, then

$$(1 \otimes m) f = 0 \implies 1m = 0 \implies m = 0,$$

i.e., $1 \otimes m = 1 \otimes 0 = 0$. Hence $\ker f = \{0\}$, so $f$ is injective. $\square$

**Remark 2.5.** From Properties 1 and 4 it follows that for any $R$-module $M$, $R \otimes M \cong M \otimes R \cong M$.

# 3 Tensor Products and Exactness

In this section we show that taking tensor product gives rise to a covariant functor from the category of $R$-modules to itself and prove that it is right exact but not left exact.

Let $\mathrm{Mod}_R$ denote the category of modules over a commutative ring $R$ and let $M$ be an $R$-module then for any object $N$ of $\mathrm{Mod}_R$, the tensor products $M \otimes N$ is an $R$-module and hence an object of the category $\mathrm{Mod}_R$. If we have a sequence of $R$-module homomorphisms

$$\begin{tikzcd} N' \arrow[r, "f"] \arrow[rr, bend right, "fg"'] & N \arrow[r, "g"] & N'', \end{tikzcd}$$

then for the identity homomorphism $1_M$ of $M$, using Proposition 2.1 and Corollary 2.2, we have the induced homomorphism of tensor products

$$\begin{tikzcd} M \otimes N' \arrow[r, "1_M \otimes f"] \arrow[rr, bend right, "(1_M \otimes f)(1_M \otimes f) = 1_M \otimes fg"'] & M \otimes N \arrow[r, "1_M \otimes g"] & M \otimes N'', \end{tikzcd}$$

and in particular for the identity homomorphism $1_N : N \longrightarrow N$ we have, in view of Corollary 2.3, that

$$M \otimes N \xrightarrow{1_M \otimes 1_N = 1_{M \otimes N}} M \otimes N.$$

Thus from what we have observed above if $F_{\otimes}^M : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ is a function which maps the object $N$ of $\mathrm{Mod}_R$ to the tensor product $M \otimes N$, i.e.,

$$F_{\otimes}^M(N) = M \otimes N,$$

and the morphism $f : N' \longrightarrow N$ between the objects $N'$ and $N$ of $\mathrm{Mod}_R$ to the morphism $1_M \otimes f$ between the corresponding objects $M \otimes N'$ and $M \otimes N$, i.e.,

$$F_{\otimes}^M(f) = 1_M \otimes f.$$

Then $F_{\otimes}^M$ satisfies all the properties of a covariant functor. The following result shows that the functor $F_{\otimes}^M$ is right exact.

**Theorem 3.1.** *Let $N, N'$ and $N''$ be $R$-modules. If*

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0, \tag{26}$$

*is an exact sequence of R-modules and homomorphisms, then corresponding sequence of tensor products*

$$M \otimes N' \xrightarrow{1_M \otimes f} M \otimes N \xrightarrow{1_M \otimes g} M \otimes N'' \longrightarrow 0, \tag{27}$$

*is exact for each R-module M.*

*Proof.* To prove that the sequence (27) is exact, we need to show that the homomorphism $1_M \otimes g$ is surjective and $\operatorname{Im} 1_M \otimes f = \ker 1_M \otimes g$.
$1_M \otimes g$ *is surjective.* To prove this it is enough to show that every generator of $M \otimes N''$ has a pre-image under $1_M \otimes g$. Let $m \otimes n'' \in M \otimes N''$, for some $m \in M$ and $n'' \in N$. Since the sequence (26) is exact, so $g$ must be surjective, therefore for the element $n''$ of $N''$ there exist some $n \in N$ such that $(n)g = n''$. Then $m \otimes n \in M \otimes N$ and we have that

$$(m \otimes n)1_M \otimes g = (m)1_M \otimes (n)g = m \otimes n'',$$

i.e., $1_M \otimes g$ is surjective.
$\operatorname{Im} 1_M \otimes f = \ker 1_M \otimes g$. Again as the sequence (26) is exact, therefore $\operatorname{Im} f = \ker g$, which implies that $fg = 0$. In view of Corollary 2.2, we have that

$$(1_M \otimes f)(1_M \otimes g) = 1_M 1_M \otimes fg = 1_M \otimes 0 = 0,$$

i.e., $\operatorname{Im} 1_M \otimes f \subseteq \ker 1_M \otimes g$, which gives rise to the natural homomorphism

$$\frac{M \otimes N}{\operatorname{Im} 1_M \otimes f} \xrightarrow{\pi} \frac{M \otimes N}{\ker 1_M \otimes g} \cong M \otimes N'' \quad (\because\ 1_M \otimes g \text{ is surjective})$$

given by

$$((x \otimes y) + \operatorname{Im} 1_M \otimes f)\,\pi = x \otimes (y)g.$$

If we can show that $\pi$ is an isomorphism, then $\operatorname{Im} 1_M \otimes f = \ker 1_M \otimes f$ and the result will follow. Define a map $\theta : M \times N'' \longrightarrow \dfrac{M \otimes N}{\operatorname{Im} 1_M \otimes f}$ by

$$(x, y'')\theta = x \otimes y + \operatorname{Im} 1_M \otimes f, \quad (x, y'') \in M \times N''$$

where $(y)g = y''$ for some $y \in N$ as $g$ is surjective. We first show that $\theta$ is well-defined. Let $y_1, y_2 \in N$ be such that $(y_1)g = (y_2)g = y''$. Then $(y_1 - y_2)g = 0$ implies that $y_1 - y_2 \in \ker g = \operatorname{Im} f$, so there exist some $y' \in N'$ such that $y_1 - y_2 = (y')f$. Then

$$x \otimes y_1 - x \otimes y_2 = x \otimes (y_1 - y_2) = x \otimes (y')f \in \operatorname{Im} 1_M \otimes f$$

$$\implies x \otimes y_1 + \operatorname{Im} 1_M \otimes f = x \otimes y_2 + \operatorname{Im} 1_M \otimes f,$$

hence the image of $(x, y'')$ under $\theta$ is independent of the choice of pre-image of $y''$ under $g$ so $\theta$ is well-defined. Verify that $\theta$ is $R$-bilinear. Now by the universal property of the tensor product $M \otimes N''$, there exist a unique $R$-module homomorphism (say) $\theta'$ such that $\theta = \phi\theta'$, i.e., the following diagram commutes

$$\begin{tikzcd}
M \times N'' \arrow[r, "\phi"] \arrow[dr, "\theta"'] & M \otimes N'' \arrow[d, "\theta'"] \\
 & \dfrac{M \otimes N}{\operatorname{Im} 1_M \otimes f},
\end{tikzcd}$$

where $\phi$ maps $(x, y'') \mapsto x \otimes y''$. Then for any $(x, y'') \in M \times N''$, we have that

$$(x, y'')\theta = (x, y'')\phi\theta' = (x \otimes y'')\theta',$$

which in view of the definition of $\theta$ implies that

$$(x \otimes y'')\theta' = x \otimes y + \operatorname{Im} 1_M \otimes f, \quad \text{where } (y)g = y'', \text{ for some } y \in N.$$

We now claim that $\pi\theta' = 1_{\frac{M \otimes N}{\operatorname{Im} 1_M \otimes f}}$ and $\theta'\pi = 1_{M \otimes N''}$. For any $(x, y) \in M \times N$, the element $x \otimes y$ is a generator of $M \otimes N$, so

$$\begin{aligned}
(x \otimes y + \operatorname{Im} 1_M \otimes f)\, \pi\theta' &= (x \otimes (y)g)\, \theta' \\
&= x \otimes y + \operatorname{Im} 1_M \otimes f \\
&= (x \otimes y + \operatorname{Im} 1_M \otimes f)\, 1_{\frac{M \otimes N}{\operatorname{Im} 1_M \otimes f}}
\end{aligned}$$

$$\implies \pi\theta' = 1_{\frac{M \otimes N}{\operatorname{Im} 1_M \otimes f}}. \tag{28}$$

As for any $(x, y'') \in M \otimes N''$, the element $x \otimes y''$ is a generator of $M \otimes N''$, therefore

$$\begin{aligned}
(x \otimes y'')\, \theta'\pi &= (x \otimes y + \operatorname{Im} 1_M \otimes f)\, \pi, \quad (y)g = y'', \text{ for some } y \in N \\
&= x \otimes (y)g \\
&= x \otimes y'' \\
&= (x \otimes y'')\, 1_{M \otimes N''}
\end{aligned}$$

$$\implies \theta'\pi = 1_{M \otimes N''}. \tag{29}$$

From (28) and (29) it now follows that $\pi$ and $\theta'$ are inverse of each other.Hence $\pi$ is an isomorphism. $\square$

**Remark 3.2. 1.** The above result says that the functor $F_{\otimes}^{M}$ is right exact. But it may not be left exact, e.g., consider the following short exact sequence of $\mathbb{Z}$-modules and homomorphisms

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0$$

where $f$ is multiplication by 2 and $g$ the canonical projection. If we tensor the above sequence with the $\mathbb{Z}$-mdoule $\mathbb{Z}_2$, then the induced sequence

$$0 \longrightarrow \mathbb{Z}_2 \otimes \mathbb{Z} \stackrel{1_{\mathbb{Z}_2} \otimes f}{\longrightarrow} \mathbb{Z}_2 \otimes \mathbb{Z} \stackrel{1_{\mathbb{Z}_2} \otimes g}{\longrightarrow} \mathbb{Z}_2 \otimes \mathbb{Z}_2 \longrightarrow 0$$

is right exact but not left exact because the homomorphism $1_{\mathbb{Z}_2} \otimes f$ is not injective since it maps every element $\bar{x} \otimes y$ of $\mathbb{Z}_2 \otimes \mathbb{Z}$ to $\bar{x} \otimes 2y = 2\bar{x} \otimes y = \bar{0} \otimes y = 0$, but $\mathbb{Z}_2 \otimes \mathbb{Z}$ is not a trivial $\mathbb{Z}$-module.

**2.** If for an $R$-module $M$, functor $F_{\otimes}^{M} : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ mapping $N$ to $M \otimes N$ is exact, i.e., both left and right exact, then $M$ is called a *flat module*, or in other words if tensoring with $M$ transforms every short exact sequences to short exact sequences, then $M$ is called a *flat module*.

# 4 Exercises

1. Show that $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d$ (as $\mathbb{Z}$-modules), where $d = \gcd(m, n)$. What happens when $m, n$ are coprime?

2. Prove that for any finite abelian group $G$, $G \otimes \mathbb{Q} = \{0\}$ as $\mathbb{Z}$-modules.

3. Let $I$ be an ideal of a commutative ring $R$ and $M$ an $R$-module. Show that $(R/I) \otimes M \cong M/IM$, where $IM$ is a submodule of $M$ consisting of all finite sums $\sum_i a_i x_i$, $a_i \in I$ and $x_i \in M$.

4. Find an example of a flat module.

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