Appendix-110
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Department of Distance and Continuing Education
University of Delhi

B.A.(Prog) Mathematics

Semester-I

Course Credit - 4

Discipline A-1

TOPICS IN CALCULUS

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Contents

Unit-1: Limits, Continuity and Differentiability

Unit Overview

The independent work of two famous mathematicians, Isaac Newton and Gottfried Leibniz. in the 17^{th} century, laid foundation to calculus. Differential Calculus and Integral Calculus are two main parts of Calculus. In this unit, we will discuss the concepts of Limits, Continuity and Differentiability. It is divided into four lessons.

In Lesson-1, we will discuss the concepts of the limit of a function. Both informal and formal approach($\epsilon - \delta$ approach) are to be used. Limits of various kinds of functions, limits at infinity and infinite limits will be discussed. Some theorems which help us to evaluate limits are to be discussed in this lesson.

In Lesson-2, we will discuss two important concepts of Calculus, namely continuity and differentiability. We will discuss important algebraic properties of these topics together with their applications. Types of discontinuity and the geometrical interpretation of the differentiability will also be discussed.

Continuing with the concepts of Lesson-1 and Lesson-2, in Lesson-3, we will discuss the higher order derivatives of functions. Leibnitz's Theorem and its uses will be discussed in this lesson.

In Lesson-4, we will discuss the partial derivatives of the functions of two and three variables. Homogeneous functions and the Euler's Theorem for homogeneous functions will be studied in this lesson.

Various examples, In-text Exercises and Self-Assessment Exercises will be included in each lesson to boost the confidence of the students.

Lesson - 1

Limits

Structure

1.1 Learning Objectives

The learning objectives of this lesson are to:

- understand the concept of the limit of a function.
- study the algebraic properties of limits.
- study the concept of infinite limits and the limits at infinity.

1.2 Introduction

The concept of a limit or limiting process is essential for the understanding of calculus. It has been around for thousands of years. In fact, early mathematicians used the limiting procedure to obtain better and accurate approximations of the areas enclosed by closed plane curves. Yet, the formal definition of a limit as we know and understand it today did not appear until the late 19th century. The idea of limits gives us a method for describing how the outputs of a function behave as the inputs approaches to some specific value. Limits are used as real-life approximations to calculate derivatives, which are helpful in finding the slope of the tangent to a curve at a point, maxima-minima of a function etc. The limit of a function is a fundamental concept in calculus and analysis concerning the behavior of that function near a particular point.

1.3 Limit (An Informal Approach)

To understand what do we mean by the limit of a function, let us first see how the function $f(x) = x^3 - 2$ behaves as the value of the variable x approaches towards 1. We have the following table:

Value of x	Value of $f(x)$	Value of x	Value of $f(x)$
0	-2	2	6
0.5	-1.875	1.5	1.375
0.9	-1.271	1.1	-.669
0.99	-1.0297	1.01	-0.969699
0.999	-1.003	1.001	-0.996997

Table 1.1: Values of $f(x) = x^3 - 2$.

We can see that the value of the function $f(x)$ approaches to -1 as the value of x approaches to 1 from both sides (left and right). This leads to the following informal definition of the limit of a function

Definition 1.1 (**Informal Definition**). A function $f(x)$ is said to have a limit L as x approaches to a, written as

$$f(x) \to L \text{ as } x \to a \quad \text{or} \quad \lim_{x \to a} f(x) = L$$

if the values of $f(x)$ can be made arbitrary close (as close as we like) to L by choosing values of x sufficiently close to a.

Note. It is to be noted that

1. the number L mentioned above is a finite real number.
2. while talking about the limit $\lim\limits_{x \to a} f(x)$, it is not necessary for the function $f(x)$ to be defined at the point $x = a$.

3. the limit defined above is also known as **two sided limit** because it requires the value of the function $f(x)$ to approach towards L as x approaches towards a from left($x < a$) and right ($x > a$). When we talk about the limit of a function at a point, we talk about the two sided limit.

Example 1.1. Consider the limit $\lim_{x \to 0} \frac{\cos x - 1}{x}$. We note that the function $\frac{\cos x - 1}{x}$ is not defined at $x = 0$, but the values of $f(x) = \frac{\cos x - 1}{x}$ approach to 0 when the value of x approach to 0, as shown in Table 1.2 and Figure 1.1.

Value of x	Value of $f(x)$	Value of x	Value of $f(x)$
-1	0.459698	1	-0.459698
-0.5	0.244835	0.5	-0.244835
-0.25	0.12435	0.25	- 0.12435
-0.1	0.049958	0.1	- 0.049958
-0.01	0.004999	0.01	-0.004999
-0.001	0.000500	0.001	-0.000500

Table 1.2: Values of $f(x) = \frac{\cos x - 1}{x}$.

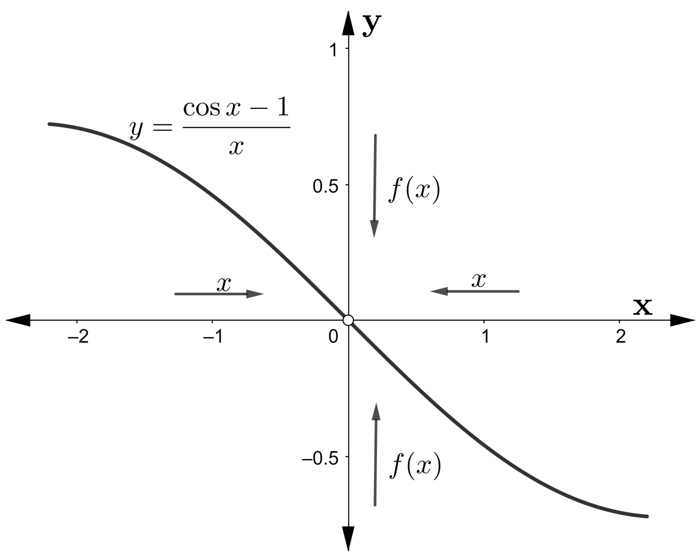


Figure 1.1: Graph of $f(x) = \frac{\cos x - 1}{x}$

Example 1.2. Let us consider the behavior of $f(x) = \sin\left(\frac{1}{x}\right)$ as $x \to 0$. From the Table 1.3 and Figure 1.2, we see that as $x \to 0$, the value of $\sin\left(\frac{1}{x}\right)$ oscillates rapidly between -1 and 1, and does not approach to a fixed real number. Therefore, in this case, we say that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Value of x	Value of $f(x)$	Value of x	Value of $f(x)$
-1	-0.841471	1	0.841471
-0.1	0.544021	0.1	-0.544021
-0.05	-0.912945	0.05	0.912945
-0.001	-0.826880	0.001	0.826880
-0.0005	-0.930040	0.0005	0.930040

Table 1.3: Values of $f(x) = \sin\left(\frac{1}{x}\right)$.

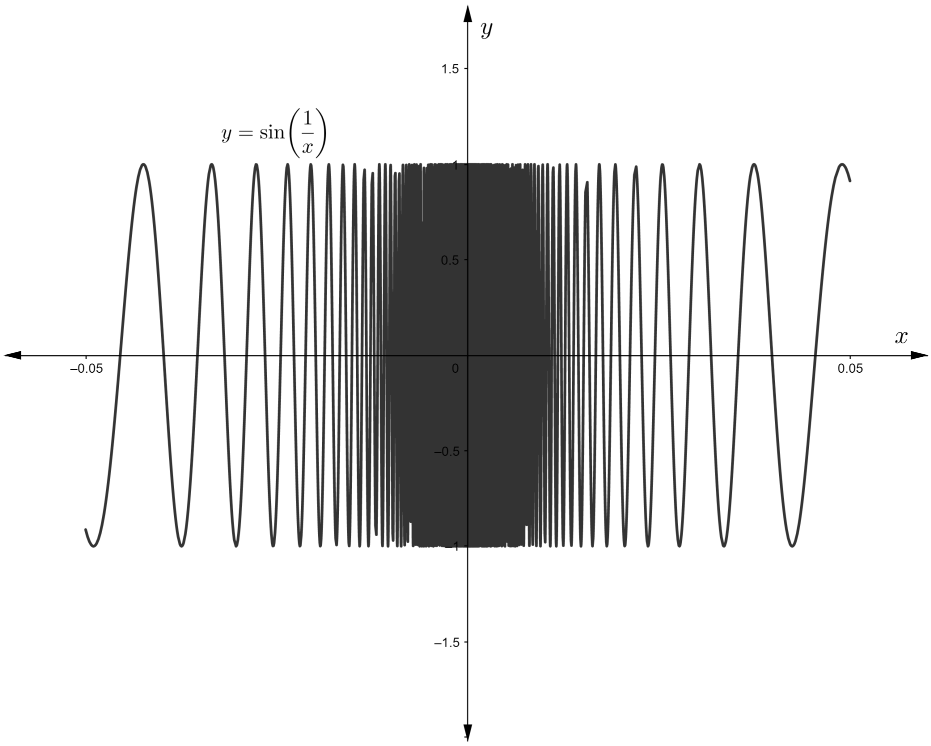


Figure 1.2: Graph of $f(x) = \sin\left(d\frac{1}{x}\right)$

Definition 1.2 (One Sided Limits).

1. A function $f(x)$ is said to have a limit L as x approaches a from the right, written $\lim_{x \to a+} f(x) = L$, if the value of $f(x)$ can be made sufficiently close (as close as we like) to L by choosing the value of x sufficiently close to a $(x > a)$. This limit is also known as the **right hand limit (R.H.L.)**.

2. A function $f(x)$ is said to have a limit L as x approaches a from the left, written $\lim_{x \to a-} f(x) = L$, if the value of $f(x)$ can be made sufficiently close (as close as we like) to L by choosing the value of x sufficiently close to a $(x < a)$. This limit is also known as the **left hand limit (L.H.L.)**.

Example 1.3. Consider the function

$$f(x) = \begin{cases} x^2 + 2, & x < 0 \\ x - 2, & x \geq 0 \end{cases}. \tag{1.1}$$

We have the following table for the values of $f(x)$ near $x = 2$:

Value of x	Value of $f(x)$	Value of x	Value of $f(x)$
-1	3	1	-1
-0.5	2.25	0.5	-1.5
-0.25	2.0625	0.25	-1.75
-0.1	2.01	0.1	-1.9
-0.01	2.0001	0.01	-1.99
-0.001	2.000001	0.001	-1.999

Table 1.4: Values of $f(x)$.

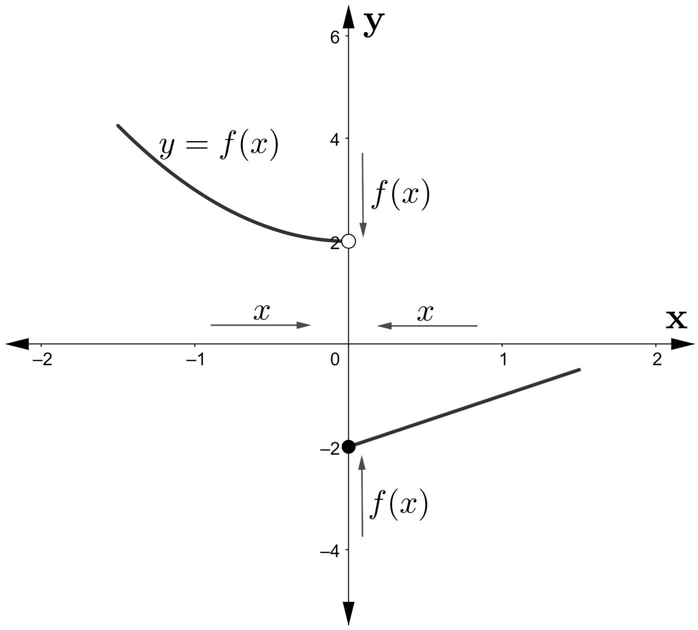


Figure 1.3: Graph of $f(x)$.

From the Table 1.4, we observe that the value of the function $f(x)$ approaches towards 2 as the value of x approaches towards 0 from the left ($x < 0$). On the other hand, if the value of x approaches towards 0 from the right ($x > 0$), the value of the function $f(x)$ approaches towards -2. Thus we write

$$\lim_{x \to 0^+} f(x) = -2 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = 2.$$

Theorem 1.1 (Necessary and Sufficient Condition). *The limit of a function $f(x)$ exists at a point $x = a$ if and only if both the two sided limits of $f(x)$ exist at $x = a$ and they are equal. That is;*

$$\lim_{x\to a} f(x) = L \quad \textit{if and only if} \quad \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x).$$

Theorem 1.2 (Non Existence of Limit). *The limit of a function $f(x)$ at $x = a$ does not exist if*

1. *Either $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ or both do not exist, or*
2. *Both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, but they are not equal.*

Definition 1.3. (**Absolute Value Function**) Let $x \in \mathbb{R}$. Then the absolute value of x is defined by the function

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}.$$

and is denoted by $|x|$.

For example,

$$|-10| = 10 \quad \text{and} \quad |10| = 10.$$

Note. We note that

1. $|0| = 0$.
2. $|x| = |-x| > 0, \quad$ for all non-zero x.

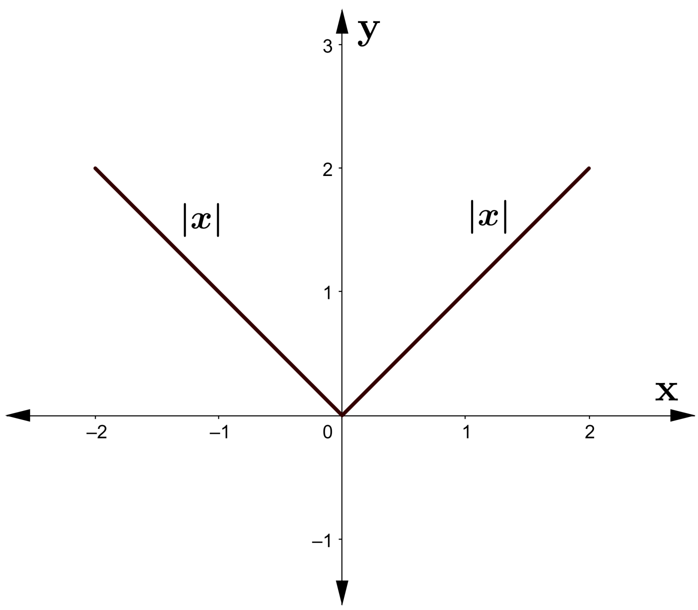


Figure 1.4: Graph of $f(x) = |x|$.

Example 1.4. Consider the function $f(x) = \lim_{x\to 0} \frac{|x|}{x}, \quad x \neq 0$. We have

$$\begin{aligned} \text{L.H.L.} &= \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0}(-1) = -1, \\ \text{and} \quad \text{R.H.L.} &= \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0}(1) = 1. \end{aligned}$$

Since, L.H.L. $\neq$ R.H.L. Therefore, $\lim_{x\to 0} \frac{|x|}{x}$ does not exists.

Definition 1.4. (**Greatest Integer Function**) Let $x \in \mathbb{R}$. Then the greatest integer function $[x]$ is defined as the largest integer less than or equal to x.
For example,

$$[2.4] = 2, \quad [2] = 2, \quad [1.9] = 1, \quad [-1] = -1 \quad \text{and} \quad [-1.3] = -2.$$

Example 1.5. Consider the function $f(x) = [x], -2 \leq x \leq 2$. We have

$$[x] = \begin{cases} -2, & -2 \leq x < -1 \\ -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & x = 2 \end{cases}$$

Now,

$$\begin{aligned} \text{L.H.L.} &= \lim_{x\to 0^-} [x] = \lim_{x\to 0}(-1) = -1, \\ \text{and} \quad \text{R.H.L.} &= \lim_{x\to 0^+} [x] = \lim_{x\to 0} 0 = 0. \end{aligned}$$

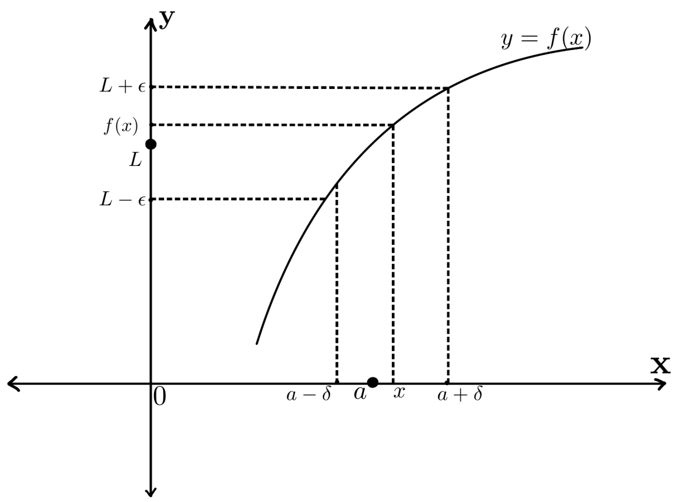
Since, L.H.L. $\neq$ R.H.L. Therefore, $\lim_{x\to 0}[x]$ does not exist. Similarly, we can show that the limit of $f(x) = [x]$ does not exist at $x = -1$ and $x = 1$.

In general, limit of $f(x) = [x]$ defined on $\mathbb{R}$ does not exit at all integer values i.e. at $x = 0, \pm 1, \pm 2, \pm 3, \ldots$.

1.4 Limit (Formal Approach)

Definition 1.5 ($\epsilon - \delta$ **Approach).** Let $f(x)$ be a real valued function defined in a set containing a, except possibly at a. Then $f(x)$ is said to approach to a real number L as x approaches to a, if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \qquad \text{when} \qquad 0 < |x - a| < \delta.$$



Note. We note the following:

1. The value of δ depends on ϵ and a.
2. For a given $\epsilon > 0$, the choice of δ is not unique. Once we find one value of δ, any value that is less than δ also works.

Example 1.6. We will show that for $a \in \mathbb{R}$, $\lim_{x \to a} c = c$, where c is a real constant.
Here $f(x) = c, L = c$.
Let $\epsilon > 0$ be an arbitrary real number. Since

$$|f(x) - L| = |c - c| = 0.$$

Let us choose any real number $\delta > 0$. So, if

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - c| = 0 < \epsilon.$$

Since $\epsilon > 0$ is an arbitrary number. Therefore, by $\epsilon - \delta$ definition, we have

$$\lim_{x \to a} c = c.$$

Example 1.7. We will show that for $a \in \mathbb{R}$, $\lim_{x \to a} x = a$. Here $f(x) = x, L = a$.
Let $\epsilon > 0$ be an arbitrary real number. Since

$$|f(x) - L| = |x - a|.$$

Let us choose $\delta = \epsilon$. Then $\delta > 0$. So, if

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| = |x - a| < \delta = \epsilon.$$

Since $\epsilon > 0$ is an arbitrary number. Therefore, by $\epsilon - \delta$ definition, we have

$$\lim_{x \to a} x = a.$$

Example 1.8. Show that for $a \in \mathbb{R}$, $\lim_{x \to a}(3x-5) = 3a-5$.

Solution. Let $f(x) = (3x-5)$ and $\epsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - (3a-5)| &= |(3x-5) - (3a-5)| \\ &= |3(x-a)| \\ &= 3|x-a| \\ &< \epsilon \quad \text{when } |x-a| < \frac{\epsilon}{3}. \end{aligned}$$

Therefore, by taking $\delta = \frac{\epsilon}{3}$, we get

$$|f(x) - (3a-5)| < \epsilon \qquad \text{when} \qquad |x-a| < \delta.$$

Hence, by $\epsilon - \delta$ definition, we have

$$\lim_{x \to a} f(x) = \lim_{x \to a}(3x-5) = 3a-5.$$

Example 1.9. Show that

$$\lim_{x \to 2}(x^2 - 3) = 1.$$

Solution. Let $f(x) = x^2 - 3$ and $\epsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - L| &= |x^2 - 3 - (1)| \\ &= |x^2 - 4| \\ &= |x-2||x+2| \\ &= |x-2|\,(|x| + |2|) \quad \text{[Using Triangle Inequality of } \mathbb{R}] \end{aligned} \tag{1.2}$$

Since, we have to find limit of $f(x)$ as x tends to 2, we take values of x near 2. Let us take x such that $|x-2| < 1$ or $1 < x < 3$. Then

$$3 < |x| + |2| < 5. \tag{1.3}$$

Therefore, from (1.2) and (1.3), we get

$$\begin{aligned} |f(x) - L| &\le |x-2|(|x| + |2|) \\ &< 5|x-2| \quad \text{when } |x-2| < 1 \\ &< \epsilon \quad \text{when } |x-2| < \frac{\epsilon}{5} \text{ as well as } |x-1| < 1 \end{aligned}$$

Therefore, by taking $\delta = \min\left\{\frac{\epsilon}{5}, 1\right\}$, we get

$$|f(x) - 1| < \epsilon \qquad \text{when} \qquad |x-2| < \delta$$

Hence, by $\epsilon - \delta$ definition, we have

$$\lim_{x \to 2} x^2 - 3 = 1.$$

In-text Exercise 1.1. Using $\epsilon - \delta$ definition, prove the following:

1. $\lim_{x\to 3} 2x + 3 = 9$
2. $\lim_{x\to 2} 3x^2 + 5 = 17$
3. $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

1.5 Algebraic Properties of Limits

Theorem 1.3. *Let*

$$\lim_{x\to a} f(x) = L \quad \textit{and} \quad \lim_{x\to a} g(x) = M,$$

then

1. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M.$
2. $\lim_{x\to a}(f-g)(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M.$
3. $\lim_{x\to a}(k\cdot f)(x) = k\cdot \lim_{x\to a} f(x) = k\cdot L,$ *where k is a real constant.*
4. $\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = L\cdot M.$
5. $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ *provided $M \neq 0$.*

Some Useful Limits:

1. $\lim_{x\to a} x^n = a^n$ for $n \in \mathbb{N}$ and for all $a \in \mathbb{R}$.
2. $\lim_{x\to a} \sin(x) = \sin(a)$, $\lim_{x\to a} \cos(x) = \cos(a)$ for all $a \in \mathbb{R}$.
3. $\lim_{x\to a} e^x = e^a$ for all $a \in \mathbb{R}$.
4. $\lim_{x\to a} \ln(x) = \ln(a)$ for all $a > 0$, where $\ln(x)$ is the natural logarithm.
5. $\lim_{x\to 0} \frac{\sin x}{x} = 1.$
6. $\lim_{x\to 0} (1+x)^{1/x} = e.$

Example 1.10.

(i) Let $p(x) = p_0 + p_1 x + \cdots + p_n x^n$ be a polynomial of degree n where $p_0, p_1, \ldots, p_n$ are constants in $\mathbb{R}$ and $p_n \neq 0$. Then, by using Theorem 1.3, we get

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a} p_0 + \lim_{x\to a} p_1 x + \cdots + \lim_{x\to a} p_n x^n \\ &= p_0 + p_1 \lim_{x\to a} x + \cdots + p_n \lim_{x\to a} x^n \\ &= p_0 + p_1 a + \cdots + p_n a^n\end{aligned}$$

That is,

$$\lim_{x\to a} p(x) = p(a)$$

(ii)
$$\begin{aligned}\lim_{x\to 1}(x\sin x + 3\ln x) &= \lim_{x\to 1}(x\sin x) + \lim_{x\to 1}(3\ln x) && \text{[Using Theorem 1.3(1)]} \\ &= \lim_{x\to 1} x \cdot \lim_{x\to 1}\sin x + 3\cdot\lim_{x\to 1}\ln x && \text{[Using Theorem 1.3(3,4)]} \\ &= 1\cdot\sin 1 + 3\cdot\ln 1 \\ &= \sin 1 + 3\cdot 0 \\ &= \sin 1 + 0 = \sin 1\end{aligned}$$

(iii)
$$\begin{aligned}\lim_{x\to 2}\frac{x^3+5}{x^2-6} &= \frac{\lim_{x\to 2}(x^3+5)}{\lim_{x\to 2}(x^2-6)} && \text{[Using Theorem 1.3(5)]} \\ &= \frac{\lim_{x\to 2} x^3 + \lim_{x\to 2} 5}{\lim_{x\to 2} x^2 - \lim_{x\to 2} 6} && \text{[Using Theorem 1.3(1)]} \\ &= \frac{\left(\lim_{x\to 2} x\right)^3 + 5}{\left(\lim_{x\to 2} x\right)^2 - 6} && \text{[Using Theorem 1.3(4)]} \\ &= \frac{2^3+5}{2^2-6} \\ &= \frac{8+5}{4-6} = \frac{13}{-2} = \frac{-13}{2}\end{aligned}$$

(iv)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{x} &= 4\cdot\lim_{4x\to 0}\frac{\sin 4x}{4x} && [x\to 0 \Longrightarrow 4x\to 0] \\ &= 4\cdot 1 \\ &= 4\end{aligned}$$

(v)
$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{\cos x}\cdot\frac{1}{x} \\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} \\ &= 1\cdot\frac{1}{1} = 1\end{aligned}$$

(vi)
$$\begin{aligned}\lim_{x\to 0}(1+2x)^{1/x} &= \lim_{2x\to 0}\left[(1+2x)^{1/2x}\right]^2\\ &= \lim_{y\to 0}\left[(1+y)^{1/y}\right]^2\\ &= \left[\lim_{y\to 0}(1+y)^{1/y}\right]^2\\ &= e^2\end{aligned}$$

In-text Exercise 1.2. Discuss the existence of the following limits:

1. $\lim_{x\to 4} f(x)$, where $f(x) = \begin{cases} x^4 - x - 1, & x \leq 4 \\ 3x - 5, & x > 4 \end{cases}$
2. $\lim_{x\to 0} \dfrac{2|x|}{|x|+5}$
3. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$
4. $\lim_{x\to 0} \dfrac{2}{(1+x)^{3/x}}$

Example 1.11.

(i) Consider the function $f(x) = \dfrac{x+2|x|}{-5x+|x|}$. We have

$$\begin{aligned}\lim_{x\to 0^-}\frac{x+2|x|}{-5x+|x|} &= \lim_{x\to 0}\frac{x-2x}{-5x-x}\\ &= \lim_{x\to 0}\frac{-x}{-6x}\\ &= \lim_{x\to 0}\frac{1}{6} = \frac{1}{6}.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{x\to 0^+}\frac{x+2|x|}{-5x+|x|} &= \lim_{x\to 0}\frac{x+2x}{-5x+x}\\ &= \lim_{x\to 0}\frac{3x}{-4x}\\ &= \lim_{x\to 0}\frac{3}{-4} = \frac{-3}{4}.\end{aligned}$$

Since, $\lim_{x\to 0^-}\dfrac{x+2|x|}{-5x+|x|} \neq \lim_{x\to 0^+}\dfrac{x+2|x|}{-5x+|x|}$. Therefore, $\lim_{x\to 0}\dfrac{x+2|x|}{-5x+|x|}$ does not exist.

(ii) Let $\dfrac{p(x)}{q(x)} = \dfrac{x^2+x-2}{x^2+2x-3}$. We note that $p(1) = 0 = q(1)$. We can write

$$\begin{aligned}\frac{p(x)}{q(x)} &= \lim_{x\to 1}\frac{x^2+x-2}{x^2+2x-3}\\ &= \lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(x+3)}\\ &= \lim_{x\to 1}\frac{(x+2)}{(x+3)}\\ &= \frac{1+2}{1+3} \quad \text{[Using Theorem 1.3(5)]}\\ &= \frac{3}{4}.\end{aligned}$$

(iii)
$$\begin{aligned}\lim_{x\to -1}\frac{x^3+1}{x+1} &= \lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{(x+1)}\\ &= \lim_{x\to 1}(x^2-x+1)\\ &= 1+1+1 \quad \text{[Using Theorem 1.3(1,4)]}\\ &= 3.\end{aligned}$$

Theorem 1.4 (Sandwich Theorem). *Let X be a subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Let $f(x), g(x)$ and $h(x)$ be functions defined on X, except possibly at $x = a$, such that*

1. $f(x) \le g(x) \le h(x) \quad \forall x \in X$
2. $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$

Then,

$$\lim_{x\to a} g(x) = L.$$

Note. The Sandwich Theorem is also known as the Squeeze Theorem.

Example 1.12. Use the Sandwich Theorem to evaluate

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$$

Solution. Let $g(x) = x\sin\left(\frac{1}{x}\right)$, $x \neq 0$. We have

$$\begin{aligned} & -1 \le \sin\left(\frac{1}{x}\right) \le 1\\ \implies \quad & -x \le x\sin\left(\frac{1}{x}\right) \le x \quad \text{for } x > 0\\ \text{and} \quad & -x \ge x\sin\left(\frac{1}{x}\right) \ge x \quad \text{for } x < 0.\end{aligned}$$

In both the cases as $\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0$, therefore, by using the Sandwich Theorem, we get

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Example 1.13. Using the inequality

$$\sin x < x < \tan x, \quad x \in \left(0, \frac{\pi}{2}\right)$$

and the Sandwich Theorem, prove that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Solution. We have

$$\begin{aligned}
& \sin x < x < \tan x, \quad x \in \left(0, \frac{\pi}{2}\right) \\
\implies \quad & \sin x < x < \frac{\sin x}{\cos x}, \quad x \in \left(0, \frac{\pi}{2}\right) \\
\implies \quad & 1 < \frac{x}{\sin x} < \frac{1}{\cos x}, \quad x \in \left(0, \frac{\pi}{2}\right) \\
\implies \quad & 1 > \frac{\sin x}{x} > \cos x, \quad x \in \left(0, \frac{\pi}{2}\right) \qquad (1.4)
\end{aligned}$$

Also, if $x \in \left(-\frac{\pi}{2}, 0\right)$ then $-x \in \left(0, \frac{\pi}{2}\right)$. Therefore, from (1.4)

$$\begin{aligned}
& 1 > \frac{\sin(-x)}{-x} > \cos(-x), \quad x \in \left(-\frac{\pi}{2}, 0\right) \\
\implies \quad & 1 > \frac{\sin x}{x} > \cos x, \quad x \in \left(-\frac{\pi}{2}, 0\right). \qquad (1.5)
\end{aligned}$$

Therefore in both cases (when $x > 0$ and when $x < 0$), we have

$$1 > \frac{\sin x}{x} > \cos x$$

and

$$\lim_{x\to 0} \cos x = 1 = \lim_{x\to 0}(1)$$

Hence, by using the Sandwich Theorem, we get

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

In-text Exercise 1.3. Find the following limits:

1. $\lim_{x\to 1} \dfrac{x^2 - x}{x^2 - 1}$.

2. $\lim_{x\to -1} \dfrac{x^2 - 1}{x^2 - 4x - 5}$.

3. Use Sandwich Theorem to prove that $\lim_{x\to 0} \sin x = 0$.

1.6 Infinite Limits and Limits at Infinity

1.6.1 Infinite Limits

1. If the values of a function $f(x)$ gets larger and larger (larger than any given $K > 0$) as the value of x approaches to a, then we say that the function $f(x)$ tends to ∞ as x tends to a and represent it by $\lim\limits_{x \to a} f(x) = \infty$.

   For example, $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$ (see Figure 1.5).

2. If the value of a function $f(x)$ gets smaller and smaller (smaller than any given $K < 0$) as the value of x approaches to a, then we say that the function $f(x)$ tends to $-\infty$ as x tends to a and represent it by $\lim\limits_{x \to a} f(x) = -\infty$.

   For example, $\lim\limits_{x \to 0} \dfrac{-1}{x^2} = -\infty$ (see Figure 1.6).

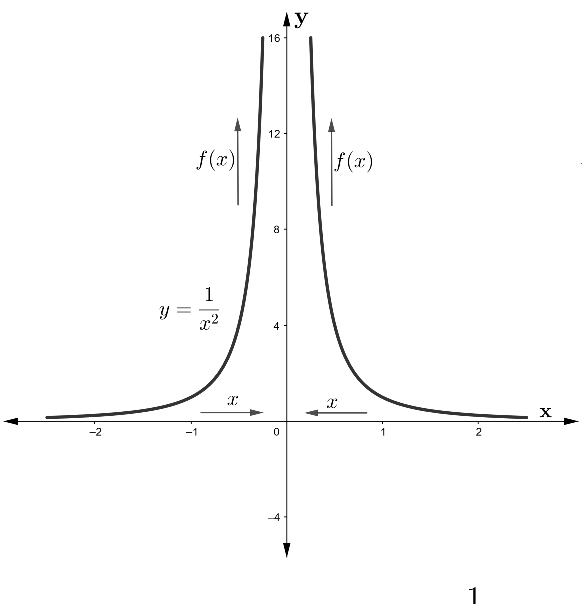


Figure 1.5: Graph of $f(x) = \dfrac{1}{x^2}$.

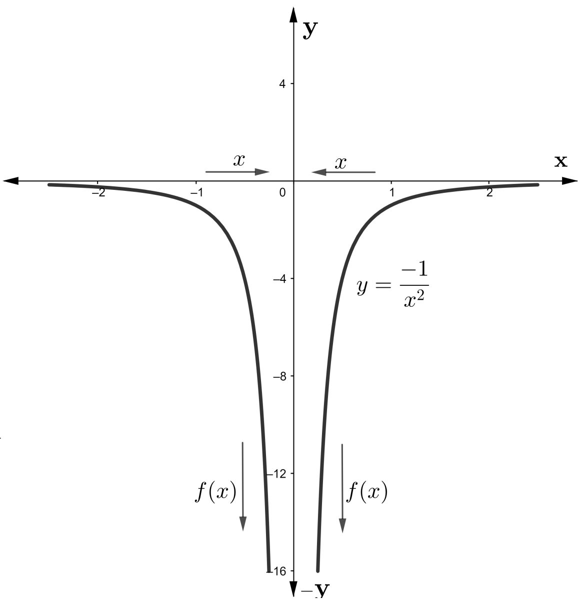


Figure 1.6: Graph of $f(x) = \dfrac{-1}{x^2}$.

3. One sided limits can be defined similarly as in section 1.3. For example, $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$ and $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$ (see Figure 1.7).

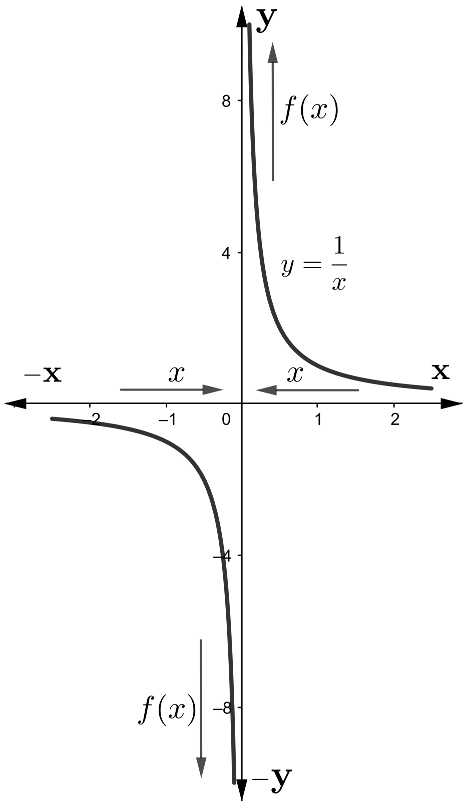


Figure 1.7: Graph of $f(x) = \dfrac{1}{x}$.

4. In the above cases 1-3, we say that $\lim\limits_{x \to a} f(x)$ does not exist, as $-\infty$ and ∞ are not fixed real numbers.

1.6.2 Limits at Infinity

1. If the values of a function $f(x)$ gets very close (as close as we like) to L as the values of x becomes larger and larger, then we say that the function $f(x)$ tends to L as x tends to ∞ and represent it by $\lim\limits_{x \to \infty} f(x) = L$.

   For example, $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$ (see Figure 1.8).

2. If the values of a function $f(x)$ gets very close (as close as we like) to L as the values of x becomes smaller and smaller, then we say that the function $f(x)$ tends to L as x tends to $-\infty$ and represent it by $\lim\limits_{x \to -\infty} f(x) = L$.

   For example, $\lim\limits_{x \to -\infty} \dfrac{-1}{x} = 0$ (see Figure 1.9).

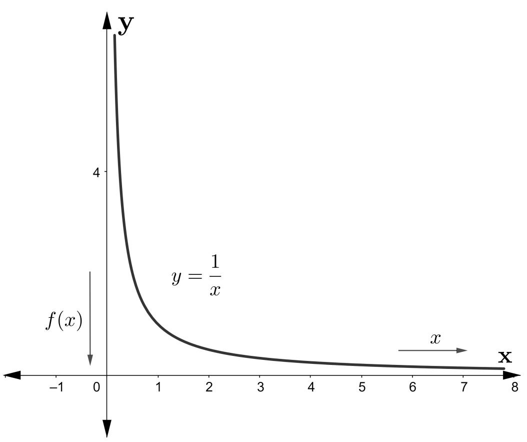


Figure 1.8: Graph of $f(x) = \frac{1}{x}$.

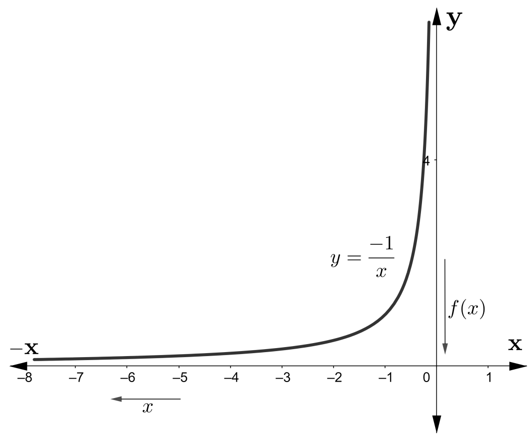


Figure 1.9: Graph of $f(x) = \dfrac{-1}{x}$.

1.6.3 Infinite Limits at Infinity

If the values of a function $f(x)$ becomes infinitely large for infinitely large/small values of x, then we say that

$$\lim_{x\to\infty} f(x) = \infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = \infty.$$

Similarly, if the values of a function $f(x)$ becomes infinitely small(negative value) for infinitely small/large values of x, then we say that

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x\to\infty} f(x) = -\infty.$$

For example, $\lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to-\infty} x^3 = -\infty$ (see Figure 1.10).

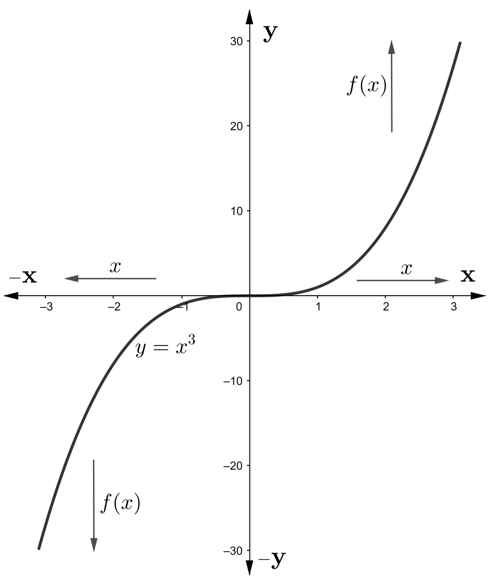


Figure 1.10: Graph of $f(x) = x^3$

Some Useful Limits:

1. $\lim\limits_{x\to a}\dfrac{1}{(x-a)^n}=\lim\limits_{x\to a^+}\dfrac{1}{(x-a)^n}=\lim\limits_{x\to a^-}\dfrac{1}{(x-a)^n}=\infty$, when n is even.
2. $\lim\limits_{x\to a^+}\dfrac{1}{(x-a)^n}=\infty$ and $\lim\limits_{x\to a^-}\dfrac{1}{(x-a)^n}=-\infty$, when n is odd.
3. $\lim\limits_{x\to\infty}x^n=\infty$ when n is even.
4. $\lim\limits_{x\to-\infty}x^n=\begin{cases}\infty, & \text{when } n \text{ is even}\\ -\infty, & \text{when } n \text{ is odd}\end{cases}.$
5. $\lim\limits_{x\to 0^+}\ln x=-\infty$
6. $\lim\limits_{x\to\infty}\ln x=\infty$
7. $\lim\limits_{x\to\infty}e^x=\infty$
8. $\lim\limits_{x\to-\infty}e^x=0$
9. If $\lim\limits_{x\to a}f(x)=\pm\infty \quad\Longrightarrow\quad \lim\limits_{x\to a}\dfrac{1}{f(x)}=0, \quad f(x)\neq 0.$
10. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$
11. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=\lim\limits_{x\to-\infty}\dfrac{\sin x}{x}=0.$

Note. Results similar to those in Theorem 1.3, can be established for the limits at infinity.

Example 1.14. Evaluate

$$\lim_{x\to 0}\frac{3-e^{2/x}}{5+e^{2/x}}.$$

Solution. We have

$$\begin{aligned} x\to 0^- &\Longrightarrow \frac{2}{x}\to-\infty\\ &\Longrightarrow e^{2/x}\to 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{L.H.L.}=\lim_{x\to 0^-}\frac{3-e^{2/x}}{5+e^{2/x}} &= \frac{\lim\limits_{x\to 0^-}(3-e^{2/x})}{\lim\limits_{x\to 0^-}(5+e^{2/x})}\\ &= \frac{3-\lim\limits_{x\to 0^-}e^{2/x}}{5+\lim\limits_{x\to 0^-}e^{2/x}}\\ &= \frac{3-0}{5+0}=\frac{3}{5}.\end{aligned}$$

Also,

$$\begin{aligned} x \to 0^+ &\implies \frac{2}{x} \to \infty \\ &\implies e^{2/x} \to \infty \\ &\implies \frac{1}{e^{2/x}} \to 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \text{R.H.L.} = \lim_{x\to 0^+} \frac{3 - e^{2/x}}{5 + e^{2/x}} &= \frac{\lim\limits_{x\to 0^+} \left(\frac{3}{e^{2/x}} - 1\right)}{\lim\limits_{x\to 0^+} \left(\frac{5}{e^{2/x}} + 1\right)} \\ &= \frac{\lim\limits_{x\to 0^+} \left(\frac{3}{e^{2/x}}\right) - 1}{\lim\limits_{x\to 0^+} \left(\frac{5}{e^{2/x}}\right) + 1} \\ &= \frac{3\cdot 0 - 1}{5\cdot 0 + 1} = -1. \end{aligned}$$

Since L.H.L.$\neq$ R.H.L., i.e. $\lim\limits_{x\to 0^-} \frac{3 - e^{2/x}}{5 + e^{2/x}} \neq \lim\limits_{x\to 0^+} \frac{3 - e^{2/x}}{5 + e^{2/x}}$. Therefore, $\lim\limits_{x\to 0} \frac{3 - e^{2/x}}{5 + e^{2/x}}$ does not exists.

Note. While computing the limits of rational functions as $x \to \pm\infty$, it is beneficial to divide the function by highest power of x that appears in the denominator. It is illustrated in the following example.

Example 1.15. Evaluate following limits:

(i) $\lim\limits_{x\to\infty} \frac{5x - 7}{x + 21}$

(ii) $\lim\limits_{x\to\infty} \frac{45x^2 - 1}{7x^4 - 11x}$

(iii) $\lim\limits_{x\to\infty} \frac{x^3 + 5x^2 - 1}{7x^2 + 21}$

Solution. (i)

$$\lim_{x\to\infty}\frac{5x-7}{x+21}=\lim_{x\to\infty}\frac{5-\dfrac{7}{x}}{1+\dfrac{21}{x}}$$
$$=\frac{\lim\limits_{x\to\infty}\left(5-\dfrac{7}{x}\right)}{\lim\limits_{x\to\infty}\left(1+\dfrac{21}{x}\right)}$$
$$=\frac{\lim\limits_{x\to\infty}5-\lim\limits_{x\to\infty}\dfrac{7}{x}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\dfrac{21}{x}}$$
$$=\frac{5-0}{1+0}\qquad\left(x\to\infty\Longrightarrow\frac{1}{x}\to 0\right)$$
$$=5.$$

(ii)

$$\lim_{x\to\infty}\frac{45x^2-1}{7x^4-11x}=\lim_{x\to\infty}\frac{\dfrac{45}{x^2}-\dfrac{1}{x^4}}{7-\dfrac{11}{x^3}}$$
$$=\frac{\lim\limits_{x\to\infty}\left(\dfrac{45}{x^2}-\dfrac{1}{x^4}\right)}{\lim\limits_{x\to\infty}\left(7-\dfrac{11}{x^3}\right)}$$
$$=\frac{\lim\limits_{x\to\infty}\dfrac{45}{x^2}-\lim\limits_{x\to\infty}\dfrac{1}{x^4}}{\lim\limits_{x\to\infty}7-\lim\limits_{x\to\infty}\dfrac{11}{x^3}}$$
$$=\frac{0-0}{7-0}\qquad\left(x\to\infty\Longrightarrow\frac{1}{x^n}\to 0\text{ for }n>0\right)$$
$$=0.$$

(iii)

$$\lim_{x\to\infty}\frac{x^3+5x^2-1}{7x^2+21}=\lim_{x\to\infty}\frac{x^3\left(1+\dfrac{5}{x^2}-\dfrac{1}{x^3}\right)}{x^2\left(7+\dfrac{21}{x^2}\right)}$$
$$=\lim_{x\to\infty}x\left(\frac{1+\dfrac{5}{x^2}-\dfrac{1}{x^3}}{7+\dfrac{21}{x^2}}\right)$$
$$=\infty$$

Example 1.16.

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+5}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+5}-x\right)\left(\frac{\sqrt{x^2+5}+x}{\sqrt{x^2+5}+x}\right)\\
&= \lim_{x\to\infty}\frac{(\sqrt{x^2+5})^2-x^2}{\sqrt{x^2+5}+x}\\
&= \lim_{x\to\infty}\frac{x^2+5-x^2}{\sqrt{x^2+5}+x}\\
&= \lim_{x\to\infty}\frac{5}{\sqrt{x^2+5}+x}\\
&= \lim_{x\to\infty}\frac{\frac{5}{x}}{\sqrt{1+\frac{5}{x^2}}+1}\\
&= \frac{\lim_{x\to\infty}\frac{5}{x}}{\sqrt{1+\lim_{x\to\infty}\frac{5}{x^2}}+1}\\
&= \frac{0}{\sqrt{1+0}+1} = 0.
\end{aligned}$$

In-text Exercise 1.4. Find the following limits:

1. $\lim_{x\to 2^+}\dfrac{x+2}{x^2-4}$.
2. $\lim_{x\to\infty}\left(\sqrt{x^4+5x^2}-x^2\right)$.
3. $\lim_{x\to\infty}\left(\sqrt{x^4+3x}-x^2\right)$.

1.7 Summary

In this lesson we have discussed the following points:

1. A function $f(x)$ is said to have a limit L as x approaches a, written $\lim_{x\to a} f(x) = L$ if the values of $f(x)$ can be made close (as close as we like) to L by choosing the values of x sufficiently close to a. It is expressed by $\lim_{x\to a} f(x) = L$.
2. A function $f(x)$ is said to have a limit L as x approaches a from the right, written $\lim_{x\to a+} f(x) = L$, if the value of $f(x)$ can be made close (as close as we like) to L by choosing values of x sufficiently close to a $(x > a)$. This limit is also known as the right hand limit (R.H.L.).
3. A function $f(x)$ is said to have a limit L as x approaches a from the left, written $\lim_{x\to a-} f(x) = L$ if the value of $f(x)$ can be made close (as close as we like) to L by

choosing values of x sufficiently close to a $(x < a)$. This limit is also known as the left hand limit (L.H.L.).

4. **Necessary and sufficient condition:** The limit of a function $f(x)$ exists at a point $x = a$ if and only if both the two sided limits of $f(x)$ exist at $x = a$ and they are equal. That is;

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x).$$

5. **Non existence of limit:** The limit of a function $f(x)$ at $x = a$ does not exist if

   (i) Either $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ or both do not exist, or

   (ii) Both $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist, but they are not equal.

6. **Formal Definition of Limit ($\epsilon - \delta$ Approach)**: Let $f(x)$ be a real valued function defined in a set containing a, except possibly at a. Then $f(x)$ is said to approach to a real number L as x approaches to a, if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \qquad \text{when} \qquad 0 < |x - a| < \delta.$$

7. **Algebra of limits:**

   (i) Limit of the sum/difference/product of functions f and g is equal to the sum/ difference/product of limits of f and g.

   (ii) Limit of the quotient of functions f and g is equal to the quotient of limits of f and g provided the limit of the divisor is not-zero.

8. **Sandwich Theorem:** Let X be a subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Let $f(x), g(x)$ and $h(x)$ be functions defined on X, except possibly at $x = a$, such that

   (i) $f(x) \leq g(x) \leq h(x) \quad \forall x \in X$

   (ii) $\lim\limits_{x\to a} f(x) = L = \lim\limits_{x\to a} h(x)$

   Then,

$$\lim_{x\to a} g(x) = L.$$

9. **Infinite limits:**

   (i) If the values of a function $f(x)$ gets larger and larger (larger than any given $K > 0$) as the value of x approaches to a, then we say that the function $f(x)$ tends to ∞ as x tends to a and represent it by $\lim\limits_{x\to a} f(x) = \infty$.

   (ii) If the value of a function $f(x)$ gets smaller and smaller (smaller than any given $K < 0$) as the value of x approaches to a, then we say that the function $f(x)$ tends to $-\infty$ as x tends to a and represent it by $\lim\limits_{x\to a} f(x) = -\infty$.

10. **Limits at Infinity:**

   (i) If the values of a function $f(x)$ gets very close (as close as we like) to L as the values of x becomes larger and larger, then we say that the function $f(x)$ tends to L as x tends to ∞ and represent it by $\lim_{x\to\infty} f(x) = L$.

   (ii) If the values of a function $f(x)$ gets very close (as close as we like) to L as the value of x becomes smaller and smaller, then we say that the function $f(x)$ tends to L as x tends to $-\infty$ and represent it by $\lim_{x\to-\infty} f(x) = L$.

11. **Infinite limits at Infinity:**

   (i) If the value of a function $f(x)$ becomes infinitely large for infinitely large/small values of x, then we say that

$$\lim_{x\to\infty} f(x) = \infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = \infty.$$

   (ii) If the value of a function $f(x)$ becomes infinitely small(negative value) for infinitely small/large values of x, then we say that

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x\to\infty} f(x) = -\infty.$$

1.8 Self-Assessment Exercises

1. Using the $\epsilon - \delta$ definition, prove the following:

(i) $\lim_{x\to 1/2}(7x-3) = 1/2$

(ii) $\lim_{x\to -2}(x^2+5) = 9$

2. Show that $\lim_{x\to 5} \frac{|x-5|}{x-5}$ does not exists.

3 . Discuss the existence of the limit of the function at $x = 0$

$$f(x) = \begin{cases} \dfrac{e^{3/x} - e^{-3/x}}{e^{3/x} + e^{-3/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

4. Find the values of a for which $\lim_{x\to 2} \frac{(x+3)(x+a)}{x^2-4}$ exists.

5. Find the value of a for which $\lim_{x\to 3} f(x)$ exists, where

$$f(x) = \begin{cases} 4x-5, & x \leq 3 \\ x+2a, & x > 3 \end{cases}.$$

6. Discuss the existence of $\lim_{x\to 0} \frac{x^2+2x}{|x|}$.

7. Find the following limits

(i) $\lim_{x\to\infty} \frac{x^4 - 3x^2 + 5}{5x^4 - 3x + 9}$,

(ii) $\lim_{x\to\infty} \frac{\sqrt{x^2 + 2}}{x^2 - 9x + 1}$,

(iii) $\lim_{x\to\infty} (\sqrt{x^8 - 4x^3} - x^4)$

(iv) $\lim_{x\to\infty} (\sqrt{x^8 - 4x^4 + 2x} - x^4)$.

8. Find the limit $\lim_{x\to 2} \left(\frac{|x-2|}{x-2} + [x] \right)$.

9. Show that $\lim_{x\to 0} \frac{x(x+2)}{|x|}$ does not exist.

10. By using Sandwich Theorem, prove that

$$\lim_{x\to 0} \cos x = 1.$$

11. Find the following limits:

(i) $\lim_{x\to\infty} \frac{\sin 2x}{\sin 5x}$

(ii) $\lim_{x\to\infty} \frac{\tan 4x}{\tan 3x}$

(iii) $\lim_{x\to\infty} \left(\frac{\sin 2x}{\tan 3x} \right)^2$

1.9 Solutions to In-text Exercises

Exercise 1.2

1. Limit does not exist.
2. 0.
3. $\frac{1}{2}$
4. $\frac{2}{e^3}$

Exercise 1.3

1. $\frac{1}{2}$.
2. $\frac{1}{3}$.

Exercise 1.4

1. Does not exist.
2. $\frac{5}{2}$.
3. 0.

1.10 Suggested Readings

1. Narayan, S. & Mittal, P. K.(2019). Differential Calculus. S. Chand Publishing.
2. Anton, H., Bivens, I. C., & Davis, S. (2015). Calculus: Early Transcendentals. John Wiley & Sons.
3. Singh, J.P. (2017). Calculus, 2nd Edition. Ane Books Pvt Ltd.

Lesson - 2

Continuity and Differentiability

Structure

2.1 Learning Objectives

The learning objectives of this lesson are to:

- learn the concepts of continuity and discontinuity of functions and their property.
- differentiate between various types of discontinuity of a function.
- learn the differentiability/derivability of functions.
- understand the geometrical interpretation of differentiation.
- calculate the derivatives of the functions of various types.

2.2 Introduction

To understand the concept of continuity, let us first consider the scenario where two people, A and B, are playing catch. When the ball leaves the hand of person A and reaches person B, we see that the ball follows an unbroken curve. It can not suddenly disappear from one point and reappear at some other point. In mathematics, these kinds of unbroken curves are known as continuous curves, and this type of curve property is called continuity. Therefore, in mathematics, a continuous function is a function such that a continuous variation (that is, a change without a jump) of the variable induces a continuous variation of the value of the function. This means that there are no sudden changes in the value (known as discontinuity).

In mathematics, the derivative of a real-valued function measures the rate of change of the function value (output value) corresponding to a change in its argument (input value). Differentiation is a fundamental tool of calculus. For example, the derivative of the displacement vector of a moving object with respect to time is the object's velocity: this measures how instantly the object's position changes when time advances. The derivative of a function of a single variable at a chosen input value, when it exists, represents geometrically the slope of the tangent line to the function's graph at that point.

2.3 Continuity

Definition 2.1. A function $f(x)$ is said to be **continuous** at a point $x = a$ if the following conditions are satisfied:

(i) $f(x)$ is defined at $x = a$.

(ii) $\lim\limits_{x \to a} f(x)$ exists.

(iii) $\lim\limits_{x \to a} f(x) = f(a)$.

Definition 2.2. A function $f(x)$ is said to be continuous on $X \subset \mathbb{R}$, if $f(x)$ is continuous at each $x \in X$.

Definition 2.3. A function $f(x)$ is said to be discontinuous at $x = a$ if it is not continuous at $x = a$, i.e., if any one of the three conditions mentioned in Definition 2.1 are not satisfied. The point $x = a$ is called a **point of discontinuity** of $f(x)$.

Example 2.1.

(i) Let $p(x), x \in \mathbb{R}$, be a polynomial function. Then $p(x)$ is a continuous function. For example, let $p(x) = 4x^2 + x - 5$. Since

$$\lim_{x \to 1}(4x^2 + x - 5) = 0 = p(1).$$

Therefore, $p(x)$ is continuous at $x = 1$.

(ii) Exponential functions are continuous on $\mathbb{R}$. Since,

$$\lim_{x\to 2} e^x = e^2, \quad \lim_{x\to 2} e^{-x} = e^{-2}, \quad \lim_{x\to 2} 3^x = 3^2 = 9.$$

Therefore, e^x, e^{-x} and 3^x are continuous at $x = 2$.

(iii) Consider the logarithmic function $f(x) = \log x, x > 0, x \in \mathbb{R}$. For $a > 0$, since we have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} \log x \\ &= \log a \\ &= f(a)\end{aligned}$$

Therefore, $f(x) = \log x$ is continuous on $(0, \infty)$. Similarly, $\ln x, \log_2 x$ are also continuous on $(0, \infty)$.

(iv) Consider the trigonometric function $f(x) = \sin x, x \in \mathbb{R}$. For $a \in \mathbb{R}$, since we have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} \sin x \\ &= \sin a \\ &= f(a)\end{aligned}$$

Therefore, $f(x) = \sin x$ is continuous on $\mathbb{R}$. Similarly, $\cos x, \tan x, \cot x, \sec x$ and $\operatorname{cosec} x$ are also continuous in their respective domains.

Example 2.2. Let

$$f(x) = \begin{cases} 5x^3 - x, & x \le 1 \\ 3x + 1, & x > 1 \end{cases}.$$

We have $f(1) = 4$, i.e., $f(x)$ is defined at $x = 1$. Now,

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} \left(5x^3 - x\right) = 5 \cdot 1^3 - 1 = 5 - 1 = 4,$$

and

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} (3x + 1) = 3 \cdot 1 + 1 = 3 + 1 = 4.$$

Therefore, L.H.L.=R.H.L. That is, $\lim_{x\to 1} f(x)$ exists and it is $4 = f(1)$. Hence, by Definition 2.1, $f(x)$ is continuous at $x = 1$.

Example 2.3. Consider the function

$$f(x) = \begin{cases} x^2 - x, & x \le 2 \\ 2x, & x > 2 \end{cases}.$$

We have $f(2) = 2$, i.e., $f(x)$ is defined at $x = 2$. Now,

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} \left(x^2 - x\right) = 2^2 - 2 = 4 - 2 = 2,$$

and

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2} (2x) = 2 \cdot 2 = 4.$$

Therefore, L.H.L.$\neq$R.H.L. That is, $\lim_{x\to 2} f(x)$ does not exists. Hence, by Definition 2.3, $f(x)$ is not continuous at $x = 2$.

In-text Exercise 2.1. Examine the continuity of the following functions at $x = 1$:

1. $f(x) = \dfrac{x^2 - 3x + 2}{x - 1}$,

2. $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$,

3. $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 1}, & x \neq 1 \\ -1, & x = 1 \end{cases}$.

Example 2.4. Consider the function f defined by

$$f(x) = \begin{cases} 2x + 5, & x < -2 \\ 1, & x = -2 \\ x + 3, & -2 < x \leq 0 \\ \sin x, & 0 < x \end{cases}$$

Let a be any real number. We will find the values of a at which $f(x)$ is continuous. Consider the following cases:
Case (i) $a < -2$: In this case, $f(x) = 2x - 5$. Since $2x - 5$ is a polynomial, therefore $f(x)$ is continuous at $x = a$.
Case (ii) $a = -2$: In this case,

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} (2x + 5) = 2(-2) + 5 = 1 = f(-2) \\ \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} (x + 3) = -2 + 3 = 1 = f(-2). \end{aligned}$$

Therefore, $\lim_{x \to -2} f(x) = 1 = f(-2)$. Hence, $f(x)$ is continuous at $a = -2$.
Case (iii) $-2 < a < 0$: In this case, $f(x) = x + 3$. Since $x + 3$ is a polynomial, therefore $f(x)$ is continuous at $x = a$.
Case (iv) $a = 0$: In this case,

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (x + 3) = 0 + 3 = 3 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \sin x = \sin 0 = 0. \end{aligned}$$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$. Therefore, $\lim_{x \to 0} f(x)$ does not exists. So, $f(x)$ is not continuous at $a = 0$.
Case (v) $a > 0$: In this case, $f(x) = \sin x$ which is continuous for all real numbers. Therefore $f(x)$ is continuous at $x = a$.

Hence $f(x)$ is continuous everywhere except at $x = 0$.

Example 2.5. Consider the function $f(x) = [x]$ (greatest integer function) defined on $[-2, 2]$ as

$$[x] = \begin{cases} -2, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$$

Since $\lim\limits_{x \to -1^-} [x] = -2 \neq -1 = \lim\limits_{x \to -1^+} [x]$, therefore $f(x) = [x]$ is not continuous at $x = -1$. Similarly, $[x]$ is not continuous at $x = 0, 1$. In general, $[x]$ over $\mathbb{R}$ is discontinuous at all integer values, i.e., $x = 0, \pm 1, \pm 2, \pm 3, \ldots$

2.3.1 Types of Discontinuity

1. **Removal Discontinuity:** A function $f(x)$ is said to have a removable discontinuity at $x = a$ if $f(x)$ is defined at $x = a$, $\lim\limits_{x \to a} f(x)$ exists but $\lim\limits_{x \to a} f(x) \neq f(a)$. Such type of discontinuity can be removed by changing the value of the function $f(x)$ at $x = a$ (see Figure 2.1). .

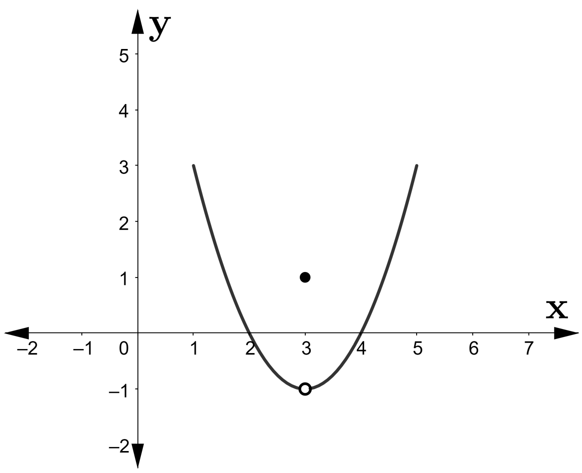


Figure 2.1: Removable discontinuity at $x = 3$.

2. **Discontinuity of First Kind:** A function $f(x)$ is said to have a discontinuity of first kind at $x = a$ if both $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exists but $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$. This is also known as **jump discontinuity** because we see a jump in the value of $f(x)$ as we cross $x = a$ from left to right or vice versa (see Figure 2.2).

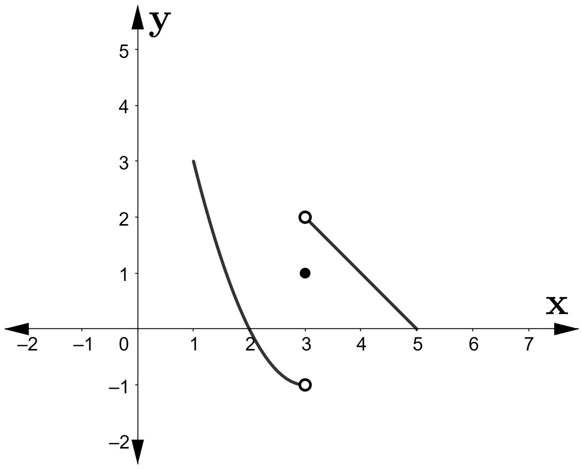


Figure 2.2: Discontinuity of first kind at $x = 3$.

(i) A function $f(x)$ is said to have a discontinuity of first kind at $x = a$ from left only, if $\lim_{x \to a^-} f(x)$ exists but $\lim_{x \to a^-} f(x) \neq f(a)$.

(ii) A function $f(x)$ is said to have a discontinuity of first kind at $x = a$ from right only, if $\lim_{x \to a^+} f(x)$ exists but $\lim_{x \to a^+} f(x) \neq f(a)$.

3. **Discontinuity of Second Kind:** A function $f(x)$ is said to have a discontinuity of second kind at $x = a$ if neither $\lim_{x \to a^+} f(x)$ nor $\lim_{x \to a^-} f(x)$ exists (see Figure 2.3).

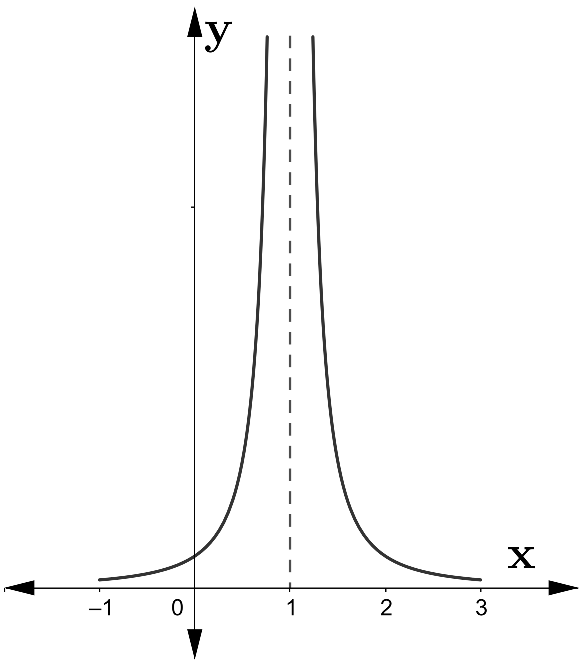


Figure 2.3: Discontinuity of second kind at $x = 1$.

(i) A function $f(x)$ is said to have a discontinuity of second kind at $x = a$ from left only, if $\lim_{x\to a^-} f(x)$ does not exists but $\lim_{x\to a^+} f(x)$ does.

(ii) A function $f(x)$ is said to have a discontinuity of first kind at $x = a$ from right only, if $\lim_{x\to a^+} f(x)$ does not exists but $\lim_{x\to a^-} f(x)$ does.

Example 2.6. Consider the following function

$$f(x) = \begin{cases} x+2, & x \neq 1 \\ 2, & x = 1 \end{cases}.$$

We note that

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+2) = 3 \neq 2 = f(1).$$

Hence $f(x)$ has a removable discontinuity at $x = 1$. The discontinuity is removed if we redefine $f(x)$ at $x = 1$ as $f(1) = 3$.

Example 2.7. Consider the following function

$$f(x) = \begin{cases} x-1, & x \leq 1 \\ 2x, & x > 1 \end{cases}.$$

We can see that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1) = 1-1 = 0$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2\cdot 1 = 2.$$

As $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$. Therefore, $f(x)$ has discontinuity of first kind at $x = 1$.

Example 2.8. Consider the following function

$$f(x) = \begin{cases} \dfrac{1}{x-1}, & x \neq 1 \\ 5x^2, & x = 1 \end{cases}.$$

We note that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x-1} = -\infty,$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x-1} = \infty.$$

Therefore, $f(x)$ has discontinuity of second kind at $x = 1$.

In-text Exercise 2.2. Examine the type of discontinuity (if any) of following functions at $x = 2$

1.
$$f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}.$$

2.
$$f(x) = \begin{cases} 3x + 5, & x < 2 \\ x^3 - 7, & x \geq 2 \end{cases}.$$

3.
$$f(x) = \begin{cases} x^2 - 1, & x \leq 2 \\ \ln(x - 2), & x > 2 \end{cases}.$$

Definition 2.4. A function $f(x)$ is said to be continuous on an open interval (a, b), if $f(x)$ is continuous at each point x such that $a < x < b$.

Note: For defining continuity on the closed interval $[a, b]$, we can not have $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$, as $f(x)$ is not defined for $x < a$ and $x > b$. Therefore, keeping this in mind, we give the following definition:

Definition 2.5. A function $f(x)$ is said to be continuous on a closed interval $[a, b]$, if

(i) $f(x)$ is continuous on (a, b),

(ii) $f(x)$ is continuous at a from the right, i.e., $\lim_{x \to a^+} f(x) = f(a)$,

(iii) $f(x)$ is continuous at b from the left, i.e., $\lim_{x \to b^-} (x) = f(b)$.

2.4 Properties of Continuous Functions

Theorem 2.1 (Algebraic Properties). *Let f and g be two functions continuous at a point $x = a$,*

$$\textit{i.e.,} \quad \lim_{x \to a} f(x) = f(a), \quad \textit{and} \quad \lim_{x \to a} g(x) = g(a),$$

then

1. *$f + g$ is continuous at $x = a$, as*

$$\lim_{x \to a}(f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a).$$

2. *$f - g$ is continuous at $x = a$, as*

$$\lim_{x \to a}(f - g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) - g(a) = (f - g)(a).$$

3. *$k \cdot f$ is continuous at $x = a$ for k (a real constant), as*

$$\lim_{x \to a}(k \cdot f)(x) = k \cdot \lim_{x \to a} f(x) = k \cdot f(a) = (k \cdot f)(a),$$

4. *$f \cdot g$ is continuous at $x = a$, as*

$$\lim_{x\to a}(f \cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = f(a) \cdot g(a) = (f \cdot g)(a),$$

5. *$\dfrac{f}{g}$ is continuous at $x = a$, provided $g(a) \neq 0$, as*

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a).$$

Theorem 2.2 (Composition of Continuous Functions). *Let $f : A \to B$ and $g : C \to D$ are two continuous functions such that $g(C) \subseteq A$, then their composition $f \circ g : C \to B$ defined by*

$$(f \circ g)(x) = f(g(x)), \quad \forall x \in C \tag{2.1}$$

is also a continuous function. Moreover,

$$\lim_{x\to a}(f \circ g)(x) = \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(g(a)) = (f \circ g)(a). \tag{2.2}$$

Example 2.9.

(i) Using Theorem 2.1, $\tan x = \dfrac{\sin x}{\cos x}$ is continuous at all points where $\cos x \neq 0$. Therefore, $\tan x$ is continuous for all $x \in \mathbb{R}\setminus\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}$. Similarly we can also find the points of continuity of other trigonometric functions.

(ii) Consider the function $f(x) = \sin\left(\frac{1}{x}\right)$. We write $f = g \circ h$, i.e., $f(x) = g(h(x))$, where

$$g(x) = \sin x, x \in \mathbb{R} \quad \text{and} \quad h(x) = \frac{1}{x}, x \neq 0.$$

Since we know that $g(x) = \sin x$ is continuous for all $x \in \mathbb{R}$ and $h(x) = \frac{1}{x}$ is continuous at all $x \in \mathbb{R}$ except at $x = 0$. Hence, by using the Theorem 2.2, $f(x) = g(h(x)) = \sin\left(\frac{1}{x}\right)$ is continuous at all $x \in \mathbb{R}$ except at $x = 0$.

(iii) Consider the function $f(x) = |\cos x|$. We write $f = g \circ h$, i.e., $f(x) = g(h(x))$, where

$$g(x) = |x|, x \in \mathbb{R} \quad \text{and} \quad h(x) = \cos x, x \in \mathbb{R}.$$

Since we know that $g(x) = |x|$ is continuous for all $x \in \mathbb{R}$ and $h(x) = \cos x$ is continuous at all $x \in \mathbb{R}$. Hence, by using the Theorem 2.2, $f(x) = g(h(x)) = |\cos x|$ is continuous at all $x \in \mathbb{R}$.

(iv) Consider the function

$$f(x) = \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Therefore, $f(x) = \text{sgn}(x)$ is continuous at all $x \in \mathbb{R}$ except at $x = 0$.

(v) Consider the function $f(x) = [\sin x]$. We write $f = g \circ h$, i.e., $f(x) = g(h(x))$, where

$$g(x) = [x], x \in \mathbb{R} \quad \text{and} \quad h(x) = \sin x, x \in \mathbb{R}.$$

Since we know that $g(x) = [x]$ is continuous for all $x \in \mathbb{R}$ except at integers, i.e., $x = 0, \pm 1, \pm 2, \ldots$ and $h(x) = \sin x$ is continuous at all $x \in \mathbb{R}$. Hence, by using the Theorem 2.2, $f(x) = g(h(x)) = [\sin x]$ is continuous at all $x \in \mathbb{R}$ except at x such that

$$\sin x = -1, 0, 1$$

$$\text{i.e.,} \qquad x = n\pi \text{ and } x = n\pi + \frac{\pi}{2} \text{ where } n = 0, \pm 1, \pm 2, \ldots$$

In-text Exercise 2.3. Find the points of discontinuity of following functions:

1. $f(x) = \operatorname{cosec} x$
2. $f(x) = \dfrac{1}{|x+1|}$
3. $f(x) = [2x]$

2.5 Differentiability

Definition 2.6. A function $f(x)$ is said to be **differentiable(derivable)** at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. This limit is known as the **derivative** of the function $f(x)$ at $x = a$ and is denoted by $f'(a)$. That is,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Note.

1. $\displaystyle\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$ is known as the **left hand derivative** of $f(x)$ at $x = a$ and it is denoted by $Lf'(a)$
2. $\displaystyle\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ is known as the **right hand derivative** of $f(x)$ at $x = a$ and it is denoted by $Rf'(a)$.
3. $f(x)$ is differentiable at the point $x = a$ if an only if $Lf'(a) = Rf'(a)$.

Example 2.10. Consider the function $f(x) = x + 2$. Let us check the differentiability of $f(x)$ at $x = 1$. We have,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h+2) - (1+2)}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

Therefore, f is differentiable at $x = 1$ and $f'(1) = 1$.

Example 2.11. Consider the function

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

. Let us check the differentiability of $f(x)$ at $x = 0$. We have,

$$Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{-h - 0}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

and

$$Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Since, $Lf'(0) \neq Rf'(0)$. Therefore $f(x)$ is not differentiable at $x = 0$.

Example 2.12. Let discuss the differentiability of $f(x) = |x| + |x - 1|$ at $x = 0$ and $x = 1$. First, we simplify the given expression of the function $f(x)$.

If $x < 0,$ then $|x| = -x$ and $|x - 1| = -(x - 1)$. Therefore $f(x) = |x| + |x - 1| = -x - (x - 1) = -2x + 1$.

If $0 \leq x < 1,$ then $|x| = x$ and $|x - 1| = -(x - 1)$. Therefore $f(x) = |x| + |x - 1| = x - (x - 1) = 1$.

If $x \geq 1,$ then $|x| = x$ and $|x - 1| = x - 1$. Therefore $f(x) = |x| + |x - 1| = x + (x - 1) = 2x - 1$.

Therefore,,

$$f(x) = \begin{cases} -2x + 1, & x < 0 \\ 1, & 0 \leq x < 1 \\ 2x - 1, & x \geq 1 \end{cases}.$$

Therefore, for the differentiability at $x = 0$, we have

$$\begin{aligned} Lf'(0) &= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{-2h+1-1}{h} \\ &= \lim_{h\to 0} \frac{-2h}{h} = -2 \\ \text{and} \qquad Rf'(0) &= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{1-1}{h} \\ &= \lim_{h\to 0} \frac{0}{h} = 0 \end{aligned}$$

Since, $Lf'(0) \neq Rf'(0)$. Therefore, $f(x)$ is not differentiable at $x = 0$.
Similarly, for the differentiability at $x = 1$, we have

$$\begin{aligned} Lf'(1) &= \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{1-1}{h} = 0 \\ \text{and} \qquad Rf'(1) &= \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{2(1+h)-1-1}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} = 2 \end{aligned}$$

Since, $Lf'(1) \neq Rf'(1)$. Therefore, $f(x)$ is not differentiable at $x = 1$.

In-text Exercise 2.4. Check the differentiability of the function

1. $f(x) = \begin{cases} x+2, & x < 2 \\ x^2, & x \geq 2 \end{cases}$, at $x = 2$.

2. $f(x) = x|x|$, at $x = 0$.

3. $f(x) = |x+1| + |x-1|$, at $x = -1$ and $x = 1$.

4. $f(x) = \begin{cases} \dfrac{\sin x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, at $x = 0$.

Definition 2.7 (Derivative Function). Let $f(x)$ be a function defined on (a, b). If $f(x)$ is derivable at each $x \in (a, b)$, then the function $f^{'}(x)$ defined by

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is known as the derivative of $f(x)$ with respect to x. It is also represented by $\frac{d}{dx} f(x) \equiv \frac{df}{dx}(x)$.

Example 2.13.

(i) Consider the constant function $f(x) = c$, where c is a real constant. The domain of $f(x)$ is $\mathbb{R}$. Therefore,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

Therefore, $f'(x) = 0$ for all $x \in \mathbb{R}$.

(ii) Consider $f(x) = x^n, x \in \mathbb{R}, n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n\right] - x^n}{h} \\
&\qquad \text{(Using Binomial Theorem)} \\
&= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n}{h} \\
&= \lim_{h \to 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} \right) \\
&= nx^{n-1}
\end{aligned}$$

Hence, $f'(x) = nx^{n-1} \, \forall \, x \in \mathbb{R}, n \in \mathbb{N}$.

(iii) Consider $f(x) = \sqrt{x}, x > 0$. Therefore,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\
&= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

Hence $f'(x) = \dfrac{1}{2\sqrt{x}}$ for $x > 0$.

Note. We note that $f'(x)$is either a constant or a function of x.

Some Useful Derivatives:

1. $\dfrac{d}{dx}e^x = e^x$ and $\dfrac{d}{dx}a^x = a^x \cdot \ln a$, where $a > 0$ is a constant.
2. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$ and $\dfrac{d}{dx}\log_a x = \dfrac{1}{x\ln a}, \quad x, a > 0.$
3. $\dfrac{d}{dx}\sin x = \cos x$ and $\dfrac{d}{dx}\cos x = -\sin x$.

2.5.1 Geometric Interpretation of a Derivative

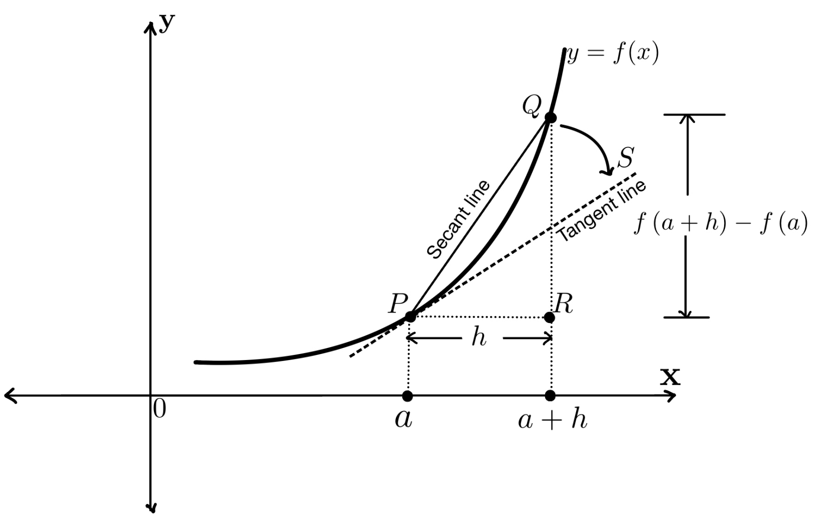


Figure 2.4: Slope of the tangent to $y = f(x)$ at the point $P(a, f(a))$.

Consider the graph of function $y = f(x)$ given in Figure 2.4. The tangent line to the graph of $y = f(x)$ at the point $P(a, f(a))$ is PS. Consider the secant lint PQ joining the points $P(a, f(a))$ and $Q(a + h, f(a + h))$. The slope of the secant line is given by

$$m_{PQ} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

From Figure 2.4 we can see that as $h \to 0$, the point Q approaches the point P along the curve and the secant line PQ approaches towards the tangent line PS. Therefore, the slope of the tangent line can be defined as the limiting case of slope of secant line. So we have

$$m_{PS} = \lim_{h\to 0} m_{PQ} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a),$$

provided the limit exists.

Therefore, the derivative $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. The equation of the tangent line at the point $(a, f(a))$ of the curve $y = f(x)$ is given by

$$y - f(a) = f^{'}(a)(x - a).$$

Also, the equation of the normal line at the point $(a, f(a))$ of the curve $y = f(x)$ is given by

$$y - f(a) = \frac{1}{f'(a)}(x - a), \text{ provided } f^{'}(a) \neq 0.$$

Example 2.14. Consider the parabola $f(x) = x^2 + 2$. We are interested in finding the equation of tangent and normal line to the given curve at the point $(2, 6)$. Since $f^{'}(x) = 2x$. Therefore, slope of the tangent at $(2, 6)$ is $f^{'}(2) = 2 \cdot 2 = 4$. Therefore, the equation of the tangent line at $(2, 6)$ is

$$\begin{aligned} & y - 6 = 4(x - 2) \\ \implies & y - 6 = 4x - 8 \\ \implies & y = 4x - 2 \end{aligned}$$

Also, the equation of the normal line at $(2, 6)$ is

$$\begin{aligned} & y - 6 = \frac{1}{4}(x - 2) \\ \implies & y - 6 = \frac{x}{4} - \frac{1}{2} \\ \implies & y = \frac{x}{4} + \frac{11}{2} \end{aligned}$$

2.5.2 Derivative as the Rate of Change

In daily life, rate of change occurs in many places. For example:

- A driver is interested in the velocity of the car which comprises of the speed he is driving with and the direction he is driving towards.

- In pandemic, the government was interested in the rate at which the Covid-19 virus was spreading relative to time, so that they can take proper measures to contain the spread.
- Scientists are interested in the rate at which the glaciers are melting with time due to temperature increase.

If $y = f(x)$ shows a relation between a variable y and x, then we define

1. The **average rate of change of y with respect to x in the interval** $[a, b]$, as
$$R_{avg} = \frac{f(b) - f(a)}{b - a}.$$
2. The **instantaneous rate of change of y with respect to x at the point** $x = a$, as
$$R_{inst} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a),$$
provided the limit exists.

2.6 Some Theorems on Derivatives

Theorem 2.3. *Every differentiable function is continuous.*

Proof. Let us assume that $f(x)$ is differentiable at $x = a$. Therefore,
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{2.3}$$
Now, we will show that $f(x)$ is continuous at $x = a$, that is
$$\lim_{x \to a} f(x) = f(a). \tag{2.4}$$
If $h = x - a$, then $x \to a$ implies $h \to 0$. Therefore from (2.4), we have
$$\lim_{h \to 0} f(a + h) = f(a). \tag{2.5}$$
Consider
$$\begin{aligned}
\lim_{h \to 0} f(a + h) &= \lim_{h \to 0} [f(a + h) - f(a) + f(a)] \\
&= \lim_{h \to 0} [f(a + h) - f(a)] + \lim_{h \to 0} f(a) \\
&= \lim_{h \to 0} \left[\frac{f(a + h) - f(a)}{h} h \right] + f(a) \\
&= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \lim_{h \to 0} h + f(a) \\
&= f'(a) \cdot 0 + f(a) \quad \text{(Using (2.3))} \\
&= f(a)
\end{aligned}$$
Therefore, $f(x)$ is continuous at $x = a$. □

Note.

1. From Theorem 2.3, we note that continuity is a necessary condition for a function to be differentiable. A function f, which is not continuous at the point $x = a$, can not be differentiable at $x = a$.

2. The converse of above theorem is not true. That is, a function which is continuous at a point may or may not be differentiable at that point. For example,

   (i) The function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

   (ii) The function $f(x) = x|x|$ is continuous as well as differentiable at $x = 0$.

Theorem 2.4. ***(Algebraic Properties)*** *Let f and g be two differentiable functions. Then*

1. $\dfrac{d}{dx}\left[f(x) + g(x)\right] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$.

2. $\dfrac{d}{dx}\left[f(x) - g(x)\right] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$.

3. $\dfrac{d}{dx}\left[k \cdot f(x)\right] = k \cdot \dfrac{d}{dx}f(x)$, *where k is a real constant.*

4. $\dfrac{d}{dx}\left[f(x) \cdot g(x)\right] = \dfrac{d}{dx}f(x) \cdot g(x) + f(x) \cdot \dfrac{d}{dx}g(x)$.

5. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\dfrac{d}{dx}f(x) \cdot g(x) - f(x) \cdot \dfrac{d}{dx}g(x)}{\left[g(x)\right]^2}$, *provided* $g(x) \neq 0$.

Example 2.15. Consider the function $f(x) = \cot x = \dfrac{\cos x}{\sin x}$. Therefore,

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(\cot x) \\
&= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{\sin x \dfrac{d}{dx}(\cos x) - \cos x \dfrac{d}{dx}(\sin x)}{\sin^2 x} \\
&= \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x} \\
&= \frac{-\left(\sin^2 x + \cos^2 x\right)}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \qquad \left(\because \sin^2 x + \cos^2 x = 1\right) \\
&= -\text{cosec}^2\, x
\end{aligned}$$

Similarly, we can prove that,

$$\frac{d}{dx}\tan x = \sec^2 x,$$
$$\frac{d}{dx}\sec x = \tan x \sec x,$$
$$\text{and}\quad \frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x.$$

Theorem 2.5. ***(Chain Rule)*** *Let $f : A \to B$ and $g : C \to D$ are two differentiable functions such that $g(C) \subseteq A$, then their composition $f \circ g : C \to B$ is also differentiable. Moreover,*

$$(f \circ g)^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x). \tag{2.6}$$

Note. Here $f^{'}(g(x))$ represents the derivative of function f with respect to taking $g(x)$ as a single variable.

Example 2.16. Consider the function $h(x) = \sin(x^2)$. If $f(x) = \sin x$ and $g(x) = x^2$, then $h(x) = f(g(x))$. So we have

$$f^{'}(x) = \cos x \Longrightarrow f^{'}(g(x)) = f^{'}(x^2) = \cos x^2,$$
$$\text{and}\quad g'(x) = 2x.$$

Therefore,

$$h'(x) = f^{'}(g(x)) \cdot g^{'}(x) = \cos x^2 \cdot 2x = 2x \cos x^2.$$

In-text Exercise 2.5. Find the derivative of the following functions:

1. $f(x) = x^2 \sin x$.
2. $f(x) = \ln(x^2 + \sin x)$.

2.6.1 Derivatives of the Inverse of an Invertible Function

Let $y = f(x)$ be an invertible differentiable function in the domain (a, b) and let $x = g(y)$ be the inverse of $y = f(x)$, i.e.,

$$(f \circ g)(y) = y \quad \text{and} \quad (g \circ f)(x) = x.$$

Therefore, we have

$$\begin{aligned} 1 &= \frac{d}{dx}(x) \\ &= \frac{d}{dx}(g \circ f)(x) \\ &= g^{'}(f(x)) \cdot f'(x) \quad \text{(Using chain rule)} \\ &= g^{'}(y) f^{'}(x) \end{aligned}$$

$$\Longrightarrow \quad g^{'}(y) = \frac{1}{f'(x)} \quad \text{or} \quad f^{'}(x) = \frac{1}{g'(y)} \tag{2.7}$$

Example 2.17. Consider the function $y = f(x) = x^{1/n}$, where $x > 0$ and $n \in \mathbb{N}$. The inverse function of $f(x)$ is given by $g(y) = y^n$. Therefore, by using (2.7), we get

$$\begin{aligned} f'(x) &= \frac{1}{g'(y)} \\ &= \frac{1}{ny^{n-1}} \\ &= \frac{1}{n} y^{1-n} \\ &= \frac{1}{n}(x^{1/n})^{1-n} \\ &= \frac{1}{n} x^{1/n-1} \end{aligned}$$

Hence,

$$\frac{d}{dx} x^{1/n} = \frac{1}{n} x^{1/n-1}, x > 0, n \in \mathbb{N}.$$

Example 2.18. Consider the function $y = f(x) = \sin^{-1}(x)$ where $x \in [-1, 1], y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. The inverse function of $f(x)$ is given by $g(y) = \sin y$. Therefore, by using (2.7), we get

$$\begin{aligned} f'(x) &= \frac{1}{g'(y)} \\ &= \frac{1}{\cos y}, \text{ provided } \cos y \neq 0 \text{ i.e., } y \neq \pm\frac{\pi}{2} \\ &= \frac{1}{\sqrt{1 - \sin^2 y}} \qquad (\because \cos y > 0 \text{ for } \frac{-\pi}{2} < y < \frac{\pi}{2}) \\ &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

Therefore,

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}, \quad x \in (-1, 1)$$

Similarly, we can also prove the following:

1. $\dfrac{d}{dx} \cos^{-1} x = \dfrac{-1}{\sqrt{1 - x^2}}, \quad \forall x \in (-1, 1)$

2. $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1 + x^2}, \quad \forall x \in \mathbb{R}$

3. $\dfrac{d}{dx} \cot^{-1} x = \dfrac{-1}{1 + x^2}, \quad \forall x \in \mathbb{R}$

4. $\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2 - 1}}, \quad \forall x \in \mathbb{R}\backslash(-1, 1)$

5. $\dfrac{d}{dx} \text{cosec}^{-1} x = \dfrac{-1}{|x|\sqrt{x^2 - 1}}, \quad \forall x \in \mathbb{R}\backslash(-1, 1)$

2.6.2 Application of Derivative

Derivatives have numerous applications. We consider the following few applications of derivatives:

1. Let $x(t)$ denote the displacement an object at time t relative to the origin. Then $v(t) = \dfrac{dx}{dt}$ gives the speed of the object at time t and $a(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ gives the acceleration of the object at time t.

2. Let $C = C(x)$ and $R = R(x)$ denote the cost function and the revenue function of a product for x units of the product. Then, $\dfrac{dC}{dx}$ and $\dfrac{dR}{dx}$ represent the marginal cost and the marginal revenue of the product.

3. For the differentiable function $y = f(x)$, $\dfrac{dy}{dx}$ is the slope of the tangent line at the point $P(x, y)$ on the curve $y = f(x)$. That is, $\dfrac{dy}{dx}$ is the slope of the curve at $P(x, y)$. Similarly, slope of the normal to the curve at $P(x, y)$ is $\dfrac{1}{\frac{dy}{dx}}$ or $\dfrac{dx}{dy}$.

4. For the function $y = f(x)$, the sign of $\dfrac{dy}{dx}$ at the point $P(a, f(a))$ determine the increasing and decreasing nature of the function at the point P.

Example 2.19. Solve the following questions:

(i) If $x(t) = t^4 - 3t^2$ represent the position of an object at time t, where t is the time in seconds and x is the displacement in meters. Find the speed and acceleration of the object at time $t = 3$ seconds.

(ii) If the total cost $C(x)$ of a commodity for x units is

$$C(x) = 5x^3 - 2x^2 - 30x + 5000$$

and the total revenue of the commodity for x units is

$$R(x) = 2 + \frac{x^3}{5}.$$

Find the marginal cost and the marginal revenue of the commodity.

(iii) Find the equation of the tangent and the normal to the curve $y = f(x) = x^2$ at the point $P(1, 1)$.

Solution. (i) We have,

$$x(t) = t^4 - 3t^2$$

$$\implies \quad v(t) = \frac{dx}{dt} = 4t^3 - 6t$$

$$\text{and} \quad a(t) = \frac{dv}{dt} = 12t^2 - 6$$

Hence, the speed of the object at $t = 3$ seconds is

$$v(3) = 4 \cdot 3^3 - 6 \cdot 3 = 90 \text{ m/sec}$$

and acceleration is

$$a(3) = 12 \cdot 3^2 - 6 = 102 \text{ m/sec}^2.$$

(ii) We have

$$\begin{aligned} & C(x) = 5x^3 - 2x^2 - 30x + 5000 \\ \Longrightarrow \quad & \text{Marginal cost} = \frac{dC}{dx} = 15x^2 - 4x - 30 \end{aligned}$$

Also,

$$\begin{aligned} & R(x) = 200 + \frac{x^3}{5} \\ \Longrightarrow \quad & \text{Marginal revenue} = \frac{dR}{dx} = \frac{3x^2}{5} \end{aligned}$$

(iii) We have

$$\begin{aligned} & y = f(x) = x^2 \\ \Longrightarrow \quad & f^{'}(x) = 2x \\ \Longrightarrow \quad & f^{'}(1) = 2 \end{aligned}$$

Therefore, the equation of the tangent line at the point $P(1, 1)$ is

$$\begin{aligned} & y - 1 = f^{'}(1)(x - 1) \\ \Longrightarrow \quad & y - 1 = 2(x - 1) \\ \Longrightarrow \quad & y = 2x - 1 \end{aligned}$$

Similarly, the equation of the normal line to the curve at the point $P(1, 1)$ is

$$\begin{aligned} & y - 1 = \frac{1}{f'(1)}(x - 1) \\ \Longrightarrow \quad & y - 1 = \frac{1}{2}(x - 1) \\ \Longrightarrow \quad & y = \frac{x}{2} + \frac{1}{2} \end{aligned}$$

2.7 Summary

In this lesson we have discussed the following points:

1. A function f is said to be **continuous** at a point $x = a$ if

(i) $f(x)$ is defined at $x = a$.

(ii) $\lim_{x\to a} f(x)$ exists.

(iii) $\lim_{x\to a} f(x) = f(a)$.

2. If any one of the three conditions mentioned above are not satisfied, then we say that the function f is **not continuous** at $x = a$ or f has **discontinuity** at $x = a$ or $x = a$ is a **point of discontinuity** of f.

3. **Type of discontinuity:**

(i) **Removal discontinuity:** A function $f(x)$ is said to have a removable discontinuity at $x = a$ if $f(x)$ is defined at $x = a$, $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq f(a)$. In this type of functions, the discontinuity can be removed by changing the value of the function $f(x)$ at $x = a$.

(ii) **Discontinuity of first kind:** A function $f(x)$ is said to have a discontinuity of first kind at $x = a$ if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists but $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$. This is also known as **jump discontinuity**, because we see a jump in the value of $f(x)$ as we cross $x = a$ from left to right or vice versa.

(iii) **Discontinuity of second kind:** A function $f(x)$ is said to have a discontinuity of second kind at $x = a$ if neither $\lim_{x\to a^+} f(x)$ nor $\lim_{x\to a^-} f(x)$ exists.

4. **Properties continuous functions:**

(i) The sum, difference and product of two continuous functions are also continuous functions.

(ii) The quotient of two continuous functions is also a continuous function provided the denominator is non-zero.

(iii) The composition of two continuous functions is also a continuous function.

5. A function $f(x)$ is said to be **differentiable/derivable** at $x = a$ if

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is known as the **derivative** of the function $f(x)$ at $x = a$ and is denoted by $f'(a)$. Therefore,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

6. **Geometric Interpretation of Differentiability:** The derivative $f^{'}(a)$ of the function $f(x)$ at $x = a$ is the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. The equation of the tangent line at point $(a, f(a))$ to the curve $y = f(x)$ is given by

$$y - f(a) = f^{'}(a)(x - a).$$

Also, the equation of the normal line at point $(a, f(a))$ to the curve $y = f(x)$ is given by

$$y - f(a) = \frac{1}{f'(a)}(x - a), \text{ provided } f'(a) \neq 0.$$

7. If $y = f(x)$ shows a relation between a variable y depending on x, then we define

   (i) The **average rate of change of** y **with respect to** x **in interval** $[a, b]$ where $h > 0$ as

   $$R_{avg} = \frac{f(b) - f(a)}{b - a}.$$

   (ii) The **instantaneous rate of change of** y **with respect to** x **at the point** $x = a$ as

   $$R_{inst} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a),$$

   provided the limit exists.

8. Every differentiable function is continuous but the converse is not necessarily true.

9. (**Algebraic Properties**) Let f and g be two differentiable functions. Then

   (i) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

   (ii) $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$.

   (iii) $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x)$, where k is a real constant.

   (iv) $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx}g(x)$.

   (v) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}f(x) \cdot g(x) - f(x) \cdot \frac{d}{dx}g(x)}{[g(x)]^2}$, provided $g(x) \neq 0$.

10. **Chain Rule:** Let $f : A \to B$ and $g : C \to D$ are two differentiable functions such that $g(C) \subseteq A$, then their composition $f \circ g : C \to B$ is also differentiable. Moreover,

    $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

11. Let $y = f(x)$ be an invertible differentiable function in the domain (a, b) and let $x = g(y)$ be the inverse of $y = f(x)$. Then

    $$g'(y) = \frac{1}{f'(x)} \quad \text{or} \quad f'(x) = \frac{1}{g'(y)}$$

12. Some applications of derivatives have been discussed.

2.8 Self-Assessment Exercises

1. Examine the continuity of the function $f(x) = |x+1| + |x-1|$ on $[-2, 2]$.

2. Obtain the points of discontinuity of the function $f(x)$ defined on $[0, 1]$ as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{2} - x, & 0 < x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ \frac{1}{2} - x, & \frac{1}{2} < x < 1 \\ 1, & x = 1 \end{cases}.$$

3. Examine the continuity at $x = 0$ and $x = 1$ of the function f defined as follows:

$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ 5x - 4, & 0 < x \leq 1 \\ 4x^2 - 3x, & x > 1 \end{cases}.$$

Also write the type of discontinuity if any.

4. Determine the values of a and b for which the function defined as follows

$$f(x) = \begin{cases} ax^2 + b, & x \leq 0 \\ 1 - \dfrac{3}{x^2+1}, & x > 0 \end{cases}$$

is continuous.

5. Examine the continuity of the function $f(x) = [x^2]$, $x \in \mathbb{R}$.

6. Discuss the differentiability/derivability of the function

$$f(x) = \begin{cases} \sqrt{x} + \dfrac{1}{x-1} & x \leq 0 \\ x^2 + \sin x & x > 0 \end{cases}$$

at $x = 0$.

7. Find the value of a and b for which the function

$$f(x) = \begin{cases} ax^2 + 2x, & x \leq 2 \\ bx - 1, & x > 2 \end{cases}$$

is differentiable at $x = 2$. (*Hint:* First find value of a and b for which it is continuous and then check for differentiability.)

8. Show that the function

$$f(x) = \begin{cases} 2x^2 + 3x - 4, & x \leq 1 \\ \sin\left(\dfrac{\pi x}{2}\right), & x > 1 \end{cases}$$

is continuous but not differentiable at the point $x = 1$.

9. Find the equation of the tangent line to the curve $y = x^3 - 3x^2 + 3$ at the point $(1, 1)$.

10. Find the equation of the normal line to the curve $y = x^2 + x\cos x + 2$ at the point $(0, 2)$.

11. Find the derivative of following functions:

(i) $f(x) = \sqrt{\cos x}$

(ii) $f(x) = \dfrac{x\sin x}{x^2 + 3x}$

(iii) $f(x) = \dfrac{\ln\sin x}{\cos x^2}$

(iv) $f(x) = e^{x^2}\sin x + x^2 e^{2x}$

(v) $f(x) = \ln\left(\dfrac{x^3 + 2x}{x^2 + 5}\right)$

12. An object is thrown from a building at a height of 128 meter above ground. The height of the object can be modeled using the position function $x(t) = 128 - 16t^2$. Find the speed and the acceleration of the object at time $t = 5$ seconds.

2.9 Solutions to In-text Exercises

Exercise 2.1

1. Not continuous at $x = 1$
2. Not continuous at $x = 1$
3. Continuous at $x = 1$

Exercise 2.2

1. No discontinuity
2. Discontinuity of first kind
3. Discontinuity of second kind

Exercise 2.3

1. $x = n\pi$ where $n = 0, \pm1, \pm2, \ldots$
2. $x = -1$
3. $x = \dfrac{n}{2}$ where $n = 0, \pm1, \pm2, \ldots$

Exercise 2.4

1. Not differentiable
2. Differentiable and $f^{'}(0) = 0$
3. Not differentiable
4. Differentiable and $f^{'}(0) = 1$

Exercise 2.5

1. $x^2 \cos x + 2x \sin x$.
2. $\dfrac{2x + \cos x}{x^2 + \sin x}$

2.10 Suggested Readings

1. Narayan, S. & Mittal, P. K.(2019). Differential Calculus. S. Chand Publishing.
2. Anton, H., Bivens, I. C., & Davis, S. (2015). Calculus: Early Transcendentals. John Wiley & Sons.
3. Singh, J.P. (2017). Calculus, 2nd Edition. Ane Books Pvt Ltd.

Lesson - 3

Successive Differentiation

Structure

3.1 Learning Objectives

The learning objectives of this lesson are to:

- understand the concept of successive differentiation.
- to calculate the n^{th} order derivatives of various functions.
- to study Leibnitz's Theorem and its applications.

3.2 Introduction

In the last lesson, we discussed the derivative of a function $f(x)$ with respect to the independent variable x. Since the derivative of $f(x)$ namely $f'(x)$ is also a function of x, we can talk about the derivative of $f'(x)$ also. In this lesson we will discuss about the higher order derivatives of $f(x)$.

Successive differentiation is the process of differentiating a given function successively and the derivatives obtained in this process are called successive derivatives.

Let $y = f(x)$ be a function of x. Then the **first derivative** of $y = f(x)$ with respect to x is denoted by $f^{'}(x)$ or $y^{'}(x)$ or y_1 or $\dfrac{dy}{dx}$ or Dy where $D \equiv \dfrac{d}{dx}$ is the differential operator.

Since $f^{'}(x) = \dfrac{dy}{dx}$ is also a function of x. If $\dfrac{dy}{dx}$ is differentiable, i.e. $y = f(x)$ is twice differentiable with respect to x, we denote its **second derivative** with respect to x by $f^{''}(x)$ or $y^{''}(x)$ or y_2 or $\dfrac{d^2y}{dx^2}$ or D^2y.

Similarly, if $\dfrac{d^2y}{dx^2}$ is differentiable, i.e. $y = f(x)$ is thrice differentiable with respect to x, we denote its **third derivative** with respect to x by $f^{'''}(x)$ or $y^{'''}(x)$ or y_3 or $\dfrac{d^3y}{dx^3}$ or D^3y.

In this manner, if $f(x)$ is differentiable n times with respect to x, we denote the $\mathbf{n^{th}}$ **derivative** of $f(x)$ with respect to x by $f^{(n)}(x)$ or $y^{(n)}(x)$ or y_n or $\dfrac{d^ny}{dx^n}$ or D^ny.

Note. We denote the n^{th} order derivative (n^{th} derivative) of the function $y = f(x)$ with respect to x at $x = a$ by $f^{(n)}(a)$ or $y^{(n)}(a)$ or $\left.\dfrac{d^ny}{dx^n}\right|_{x=a}$.

Example 3.1. Consider the function $f(x) = x\cos 2x + e^{2x}$. We have

$$\begin{aligned}
f'(x) &= \cos 2x - 2x\sin 2x + 2e^{2x},\\
f^{''}(x) &= -2\sin 2x - 2\sin 2x - 4x\cos 2x + 4e^{2x} \quad \text{(By using the chain rule)}\\
&= -4\sin 2x - 4x\cos 2x + 4e^{2x},\\
\text{and } f^{'''}(x) &= -8\cos 2x - 4\cos 2x + 8\sin 2x + 8e^{2x}\\
&= -12\cos 2x + 8\sin 2x + 8e^{2x}.
\end{aligned}$$

Example 3.2. Consider the function $y = \dfrac{\ln x}{x}$. Let us calculate the second order derivative $\dfrac{d^2y}{dx^2}$. We have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} \\
&= \frac{1 - \ln x}{x^2} \\
\Longrightarrow \quad \frac{d^2y}{dx^2} &= \frac{x^2 \cdot \frac{-1}{x} - (1 - \ln x) \cdot 2x}{x^4} \\
&= \frac{-x - 2x\,(1 - \ln x)}{x^4} \\
&= \frac{-1 - 2\,(1 - \ln x)}{x^3} \\
\Longrightarrow \quad \frac{d^2y}{dx^2} &= \frac{2\ln x - 3}{x^3}
\end{aligned}$$

Example 3.3. Let $y = x + \tan x$. We will prove that

$$\cos^2 x \frac{d^2y}{dx^2} + -2y + 2x = 0.$$

First we calculate the derivatives of y as desired.

$$\begin{aligned}
\frac{dy}{dx} &= 1 + \sec^2 x \\
\text{and} \qquad \frac{d^2y}{dx^2} &= 2\sec x \cdot \sec x \tan x \quad \text{(By using the chain rule)} \\
&= 2\sec^2 x \tan x
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\cos^2 x \frac{d^2y}{dx^2} + -2y + 2x &= \cos^2 x \cdot 2\sec^2 x \tan x - 2(x + \tan x) + 2x \\
&= 0.
\end{aligned}$$

Example 3.4. Let $x = a(t + \sin t)$ and $y = a(1 + \cos t)$. We will find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$. We have

$$\begin{aligned}
x = a(t + \sin t) &\Longrightarrow \frac{dx}{dt} = a(1 + \cos t) \\
y = a(1 + \cos t) &\Longrightarrow \frac{dy}{dt} = -a \sin t
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx}\\ &= \frac{dy/dt}{dx/dt}\\ &= \frac{-a\sin t}{a(1+\cos t)}\\ &= \frac{-2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}\\ &= -\tan\frac{t}{2}\end{aligned}$$

Now

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}\\ &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}\\ &= \frac{\frac{-1}{2}\sec^2\frac{t}{2}}{a(1+\cos t)}\\ &= \frac{\frac{-1}{2}\sec^2\frac{t}{2}}{2a\cos^2\frac{t}{2}}\\ &= \frac{-1}{4a}\sec^4\frac{t}{2}\end{aligned}$$

Therefore,

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = \frac{-1}{4a}\sec^4\frac{\pi}{4} = \frac{-1}{4a}(\sqrt{2})^4 = \frac{-1}{a}$$

In-text Exercise 3.1. Solve the following questions:

1. Find $f'''(x)$ where $f(x) = \sin(2x^3)$.

2. If $y(x) = a\cos 2x + b\sin 2x$, where a and b are real constants, then show that $y'' + 4y = 0$.

3. If $x = \sin t, y = \sin at$, show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0.$$

4. If $y = \sin(\sin x)$,show that

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0.$$

3.3 n$^{\text{th}}$ Derivatives of Some Standard Functions

3.3.1 n$^{\text{th}}$ Derivative of $(ax+b)^m$, where a and b are Constants

Let $y = (ax+b)^m$. Differentiating with respect to x successively, we get

$$\begin{aligned}
y_1 &= ma(ax+b)^{m-1}\\
y_2 &= m(m-1)a^2(ax+b)^{m-2}\\
y_3 &= m(m-1)(m-2)a^3(ax+b)^{m-3}\\
&\vdots\\
y_n &= m(m-1)(m-2)\cdots(m-(n-1))a^n(ax+b)^{m-n} \qquad (3.1)
\end{aligned}$$

Case 1. m is a positive integer such that $m \geq n$. Then from (3.1), we have

$$y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

where $m! = m(m-1)(m-2)\cdots 3\cdot 2\cdot 1$. Therefore,

$$\boxed{\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}, \quad m \geq n, m > 0} \qquad (3.2)$$

This also implies that the m$^{\text{th}}$ order derivative of $(ax+b)^m$ is $m!a^m$, which is a constant. That is

$$\boxed{\frac{d^m}{dx^m}(ax+b)^m = m!a^m} \qquad (3.3)$$

Case 2. If $m = -1$, then $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$. Then from (3.1), we have

$$\begin{aligned}
y_n &= (-1)(-2)(-3)\cdots(-1-(n-1))a^n(ax+b)^{-1-n}\\
&= (-1)(-2)\cdots(-n)a^n(ax+b)^{-(n+1)}\\
&= \frac{(-1)^n(n!)a^n}{(ax+b)^{n+1}}
\end{aligned}$$

Therefore,

$$\boxed{\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = \frac{(-1)^n\,(n!)\,a^n}{(ax+b)^{n+1}}} \qquad (3.4)$$

Example 3.5. Find the 8$^{\text{th}}$ derivative of $y = (8x+7)^{10}$.

Solution. We have,

$$y = (8x+7)^{10}$$

Therefore, by using (3.2), with $m = 10$ and $n = 8$, we get

$$y_8 = \frac{10!}{2!}\, 8^8\, (8x+7)^2.$$

Example 3.6. Find the n$^{\text{th}}$ derivative of the function

$$y = \frac{1}{(x-1)^2}.$$

Solution. We have

$$\begin{aligned} y &= \frac{1}{(x-1)^2} \\ &= -\frac{d}{dx}\left(\frac{1}{x-1}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{n}^{\text{th}} \text{ derivative of } y &= -(\text{n}+1)^{\text{th}} \text{ derivative of } \frac{1}{x-1} \\ &= -\frac{d^{n+1}}{dx^{n+1}}\left(\frac{1}{x-1}\right) \\ &= -\frac{(-1)^{n+1}\,(n+1)!\,1^{n+1}}{(x-1)^{n+2}} \\ &= \frac{(-1)^{n+2}\,(n+1)!}{(x-1)^{n+2}} \end{aligned}$$

3.3.2 $\mathbf{n^{th}}$ Derivative of $\ln(ax+b)$, where a and b are Constants

Let $y = \ln(ax+b)$. We have

$$y_1 = \frac{a}{ax+b}.$$

Since, $y_n = (n-1)^{th}$ derivative of y_1, therefore by using (3.4), we have

$$y_n = a \cdot \frac{(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n} = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

. Therefore,

$$\boxed{\frac{d^n}{dx^n}\ln(ax+b) = \frac{(-1)^{n-1}\,(n-1)!\,a^n}{(ax+b)^n}} \tag{3.5}$$

Example 3.7. Consider the function $y = \ln(4x-3)$. Then, by using (3.5), the n$^{\text{th}}$ derivative of y is

$$y_n = \frac{(-1)^{n-1}(n-1)!4^n}{(4x-3)^n}.$$

3.3.3 n$^{\text{th}}$ Derivative of a^{mx}, where a and m are Constants

Let $y = a^{mx}$. Differentiating successively with respect to x, we get

$$\begin{aligned} y_1 &= m a^{mx} \ln a, \\ y_2 &= m \ln a \cdot m a^{mx} \ln a = m^2 (\ln a)^2 a^{mx}, \\ y_3 &= m^2 (\ln a)^2 \cdot m a^{mx} \ln a = m^3 (\ln a)^3 a^{mx}, \\ &\vdots \\ y_n &= m^n (\ln a)^n a^{mx}. \end{aligned}$$

Therefore,

$$\boxed{\frac{d^n}{dx^n} a^{mx} = m^n (\ln a)^n a^{mx}} \tag{3.6}$$

If $a = e$, then $\ln e = 1$. Therefore,

$$\boxed{\frac{d^n}{dx^n} e^{mx} = m^n e^{mx}} \tag{3.7}$$

Example 3.8. Consider the function $y = 5^{3x} + e^{2x}$. Then, by using (3.6) and (3.7), the n$^{\text{th}}$ derivative of y is

$$y_n = 3^n (\ln 3)^n 5^{3x} + 2^n e^{2x}.$$

3.3.4 n$^{\text{th}}$ Derivative of $\sin(ax + b)$, where a and b are Constants

Let $y = \sin(ax + b)$. Differentiating successively with respect to x, we get

$$\begin{aligned} y_1 &= a \cos(ax + b) = a \sin\left(\frac{\pi}{2} + ax + b\right) \quad \left(\because \sin\left(\frac{\pi}{2} + \theta\right) = \sin\theta\right), \\ y_2 &= a \cdot a \cos\left(\frac{\pi}{2} + ax + b\right) = a^2 \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right) = a^2 \sin\left(\frac{2\pi}{2} + ax + b\right), \\ y_3 &= a^2 \cdot a \cos\left(\frac{2\pi}{2} + ax + b\right) = a^3 \sin\left(\frac{3\pi}{2} + ax + b\right), \\ &\vdots \\ y_n &= a^n \sin\left(\frac{n\pi}{2} + ax + b\right). \end{aligned}$$

Therefore,

$$\boxed{\frac{d^n}{dx^n} \sin(ax + b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)} \tag{3.8}$$

3.3.5 $\mathrm{n^{th}}$ Derivative of $\cos(ax+b)$, where a and b are Constants

Let $y = \cos(ax+b)$. Differentiating successively with respect to x, we get

$$y_1 = -a\sin(ax+b) = a\cos\left(\frac{\pi}{2}+ax+b\right) \quad \left(\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right),$$
$$y_2 = a\cdot(-a)\sin\left(\frac{\pi}{2}+ax+b\right) = a^2\cos\left(\frac{\pi}{2}+\frac{\pi}{2}+ax+b\right) = a^2\cos\left(\frac{2\pi}{2}+ax+b\right),$$
$$y_3 = a^2\cdot(-a)\sin\left(\frac{2\pi}{2}+ax+b\right) = a^3\cos\left(\frac{3\pi}{2}+ax+b\right),$$
$$\vdots$$
$$y_n = a^n\cos\left(\frac{n\pi}{2}+ax+b\right).$$

Therefore,

$$\boxed{\frac{d^n}{dx^n}\cos(ax+b) = a^n\cos\left(\frac{n\pi}{2}+ax+b\right)} \tag{3.9}$$

Example 3.9. Consider the function $y = \sin 2x \sin x$. To find the $\mathrm{n^{th}}$ derivative, we proceed as

$$\begin{aligned} y &= \sin 2x\sin x \\ &= \frac{1}{2}\cdot 2\sin 2x\sin x \\ &= \frac{1}{2}[\cos(2x-x)-\cos(2x+x)] \quad (\because \cos(x-y)-\cos(x+y) = 2\sin x\sin y) \\ &= \frac{1}{2}[\cos x - \cos 3x] \end{aligned}$$

Therefore using (3.9), we have

$$y_n = \frac{1}{2}\left[\cos\left(\frac{n\pi}{2}+x\right) - 3^n\cos\left(\frac{n\pi}{2}+3x\right)\right].$$

Example 3.10. Consider the function $y = \cos^2 x\sin x$. To find the $\mathrm{n^{th}}$ derivative, we proceed as

$$\begin{aligned} y &= \frac{1}{2}\cdot 2\cos^2 x\sin x \\ &= \frac{1}{2}(1+\cos 2x)\sin x \quad \left(\because \cos 2x = 2\cos^2 x - 1\right) \\ &= \frac{1}{2}[\sin x + \cos 2x\sin x] \\ &= \frac{1}{2}\sin x + \frac{1}{2}\cos 2x\sin x \\ &= \frac{1}{2}\sin x + \frac{1}{4}\cdot 2\cos 2x\sin x \\ &= \frac{1}{2}\sin x + \frac{1}{4}[\sin 3x - \sin x] \quad (\because \sin(x+y)+\sin(x-y) = 2\sin x\cos y) \\ &= \frac{1}{4}\sin x + \frac{1}{4}\sin 3x \end{aligned}$$

By using (3.8), we have

$$y_n = \frac{1}{4}\sin\left(\frac{n\pi}{2} + x\right) + \frac{3^n}{4}\sin\left(\frac{n\pi}{2} + 3x\right)$$

3.3.6 $\mathrm{n^{th}}$ Derivative of $e^{ax}\sin(bx+c)$, where a, b and c are Constants

Let $y = e^{ax}\sin(bx+c)$. Differentiating with respect to x, we get

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c) = e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right] \quad (3.10)$$

To formulate a proper generalization, let us assume

$$a = r\cos\theta, \quad b = r\sin\theta \quad (3.11)$$

where $r \geq 0$. Therefore, we have

$$\begin{aligned} a^2 + b^2 &= r^2\cos^2\theta + r^2\sin^2\theta = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 \\ \implies \quad r &= \sqrt{a^2+b^2} \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \frac{b}{a} &= \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta \\ \implies \quad \theta &= \tan^{-1}\left(\frac{b}{a}\right). \end{aligned} \quad (3.13)$$

Therefore, using (3.11), we can write (3.10) in the form

$$\begin{aligned} y_1 &= e^{ax}\left[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)\right] \\ &= re^{ax}\sin(bx+c+\theta) \quad (\because \sin(x+y) = \sin x\cos y + \cos x\sin y) \end{aligned}$$

We can see that y_1 is obtained by multiplying y by r and increasing $bx+c$ by the constant θ. Using the same process we can calculate y_2 from y_1 and so on. Hence, we have

$$\begin{aligned} y_2 &= r^2e^{ax}\sin(bx+c+2\theta), \\ y_3 &= r^3e^{ax}\sin(bx+c+3\theta), \\ &\vdots \\ y_n &= r^ne^{ax}\sin(bx+c+n\theta). \end{aligned}$$

Therefore,

$$\boxed{\frac{d^n}{dx^n}e^{ax}\sin(bx+c) = r^ne^{ax}\sin(bx+c+n\theta), \text{ where } r = \sqrt{a^2+b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)} \quad (3.14)$$

3.3.7 $\mathrm{n^{th}}$ Derivative of $e^{ax}\cos(bx+c)$, where a, b and c are Constants

Let $y = e^{ax}\cos(bx+c)$. Differentiating with respect to x, we get

$$y_1 = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c) = e^{ax}\left[a\cos(bx+c) + b\sin(bx+c)\right] \tag{3.15}$$

To formulate a proper generalization, let us assume

$$a = r\cos\theta, \quad b = r\sin\theta \tag{3.16}$$

where $r \geq 0$. Therefore, we have

$$r = \sqrt{a^2+b^2}, \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

Therefore, using (3.16), we can write (3.15) in the form

$$\begin{aligned} y_1 &= e^{ax}\left[r\cos\theta\cos(bx+c) - r\sin\theta\sin(bx+c)\right] \\ &= re^{ax}\cos(bx+c+\theta) \quad (\because \cos(x+y) = \cos x\cos y - \sin x \sin y) \end{aligned}$$

We can see that y_1 is obtained by multiplying y by r and increasing $bx+c$ by the constant θ. Using the same process we can calculate y_2 from y_1 and so on.
Hence, we have

$$\begin{aligned} y_2 &= r^2e^{ax}\cos(bx+c+2\theta), \\ y_3 &= r^3e^{ax}\cos(bx+c+3\theta), \\ &\vdots \\ y_n &= r^ne^{ax}\cos(bx+c+n\theta). \end{aligned}$$

Therefore,

$$\boxed{\frac{d^n}{dx^n}e^{ax}\cos(bx+c) = r^ne^{ax}\cos(bx+c+n\theta), \text{ where } r = \sqrt{a^2+b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)} \tag{3.17}$$

Example 3.11. Consider the function $y = e^{2x}\sin(2x-1)$. Comparing it with the function $y = e^{ax}\sin(bx+c)$, we have $a = 2, b = 2$ and $c = -1$. Therefore,

$$\begin{aligned} r &= \sqrt{a^2+b^2} = \sqrt{2^2+2^2} = \sqrt{8}, \\ \text{and} \qquad \theta &= \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}. \end{aligned}$$

Therefore, using (3.14), the $\mathrm{n^{th}}$ derivative of y is

$$\begin{aligned} y_n &= r^ne^{2x}\sin(2x-1+n\theta) \\ &= (\sqrt{8})^ne^{2x}\sin\left(2x-1+n\frac{\pi}{4}\right). \end{aligned}$$

Example 3.12. Consider the function $y = e^{2x}\sin 2x \sin x$. To find the n^th^ derivative, we proceed as

$$\begin{aligned} y &= e^{2x}\sin 2x \sin x \\ &= \frac{e^{2x}}{2}\left(2\sin 2x \sin x\right) \\ &= \frac{e^{2x}}{2}\left(\cos x - \cos 3x\right) \\ &= \frac{1}{2}\left(e^{2x}\cos x - e^{2x}\cos 3x\right) \end{aligned}$$

Therefore, using (3.17), the n$^{\text{th}}$ derivative of y is

$$y_n = \frac{1}{2}\left[r_1^n e^{2x}\cos\left(x + n\theta_1\right) - r_2^n e^{2x}\cos\left(3x + n\theta_2\right)\right] \tag{3.18}$$

where

$$\begin{aligned} r_1 &= \sqrt{2^2+1^2} = \sqrt{5}, \\ \theta_1 &= \tan^{-1}\left(\frac{1}{2}\right), \\ r_2 &= \sqrt{2^2+3^2} = \sqrt{13}, \\ \text{and} \qquad \theta_2 &= \tan^{-1}\left(\frac{3}{2}\right). \end{aligned}$$

In-text Exercise 3.2. Find the n$^{\text{th}}$ derivative of following functions:

1. $y = (5x-6)^n$.
2. $y = \dfrac{3}{(x+2)^2}$.
3. $y = e^{2x+3}$.
4. $y = \cos^2 x$.
5. $y = e^{2x}\cos 3x$.

3.4 n$^{\text{th}}$ Derivative of Rational Functions

In order to find the n$^{\text{th}}$ derivative of a rational function, we use method of partial fractions. If the denominator of each partial fraction obtained consists of real linear factors we use formulae derived in the previous section to find the n$^{\text{th}}$ derivative of the given rational function. We can take help of following table to calculate partial fractions:

S.N.	Rational Fraction	Partial Fraction Form
1.	$\dfrac{ax+b}{(x-p)(x-q)}$	$\dfrac{A}{x-p}+\dfrac{B}{x-q}$
2.	$\dfrac{ax+b}{(x-p)^2}$	$\dfrac{A}{x-p}+\dfrac{B}{(x-p)^2}$
3.	$\dfrac{ax^2+bx+c}{(x-p)(x-q)(x-r)}$	$\dfrac{A}{x-p}+\dfrac{B}{x-q}+\dfrac{C}{x-r}$
4.	$\dfrac{ax^2+bx+c}{(x-p)^2(x-q)}$	$\dfrac{A}{x-p}+\dfrac{B}{(x-p)^2}+\dfrac{C}{x-q}$
5.	$\dfrac{ax^2+bx+c}{(x-p)(x^2+qx+r)}$	$\dfrac{A}{x-p}+\dfrac{Bx+C}{x^2+qx+r}$

where A, B, C are constants in all the above partial fractions.

Example 3.13. Consider the function $y = \dfrac{x}{(x+3)(2x+5)}$. Now, to resolve it into partial fractions, let

$$\frac{x}{(x+3)(2x+5)} = \frac{A}{x+3} + \frac{B}{2x+5}, \quad (A, B \text{ are constants}) \tag{3.19}$$

$$\implies \frac{x}{(x+3)(2x+5)} = \frac{A(2x+5)+B(x+3)}{(x+3)(2x+5)}$$

$$\implies x = A(2x+5)+B(x+3)$$

$$\implies x = (2A+B)x + 5A + 3B$$

By equating coefficients of x and the constant terms on both sides, we get

$$\begin{aligned} 2A+B &= 1 \\ 5A+3B &= 0 \end{aligned} \tag{3.20}$$

By solving (3.20), we get $A = 3, B = -5$. Substituting values of A and B in (3.19), we have

$$y = \frac{x}{(x+3)(2x+5)} = \frac{3}{x+3} - \frac{5}{2x+5}$$

Hence, by using (3.4), we have

$$\begin{aligned} y_n &= 3 \cdot \frac{(-1)^n\, n!\, 1^n}{(x+3)^{n+1}} - 5 \cdot \frac{(-1)^n\, n!\, 2^n}{(2x+5)^{n+1}} \\ &= (-1)^n\, n! \left[\frac{3}{(x+3)^{n+1}} - \frac{5 \cdot 2^n}{(2x+5)^{n+1}} \right]. \end{aligned}$$

Example 3.14. Consider the function $y = \dfrac{4x}{(x-3)^2(x+1)}$. Now, to resolve it into partial fractions, let

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}, \quad (A, B, C \text{ are constants}) \tag{3.21}$$

$$\implies \frac{4x}{(x-1)^2(x+1)} = \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$$\implies 4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$\implies 4x = (A+C)x^2 + (B-2C)x - A + B + C$$

By equating coefficients of x^2, x and the constant terms on both sides, we get

$$\begin{aligned} A + C &= 0 \\ B - 2C &= 4 \\ -A + B + C &= 0 \end{aligned} \tag{3.22}$$

By solving (3.22), we get $A = 1, B = 2, C = -1$. Substituting values of A, B and C in (3.21), we have

$$y = \frac{4x}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

Hence, by using (3.4), we have

$$\begin{aligned} y_n &= \frac{(-1)^n\, n!\, 1^n}{(x-1)^{n+1}} + 2 \cdot \frac{(-1)^{n+2}\,(n+1)!\, 1^n}{(x-1)^{n+2}} - \frac{(-1)^n\, n!\, 1^n}{(x+1)^{n+1}} \\ &= (-1)^n\, n! \left[\frac{1}{(x-1)^{n+1}} + \frac{2(n+1)}{(x-1)^{n+2}} - \frac{1}{(x+1)^{n+1}}\right] \end{aligned}$$

Note. If the denominator of the given rational function is not resolvable into real linear factors, then n$^{\text{th}}$ derivative is calculated with the help of the **De Moivre's theorem**, which states that

$$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta, \text{ where } i = \sqrt{-1},$$

and by using the factorization $x^2 + a^2 = (x+ia)(x-ia)$.

Example 3.15. Consider the function $y = \dfrac{x}{x^2+a^2}$. We can write

$$\begin{aligned} y &= \frac{x}{x^2+a^2} \\ &= \frac{x}{(x+ia)(x-ia)} \\ &= \frac{1}{2}\left[\frac{1}{x+ia} + \frac{1}{x-ia}\right] \quad \text{(Using partial fractions)} \\ \implies y_n &= \frac{(-1)^n\, n!}{2}\left[\frac{1}{(x+ia)^{n+1}} + \frac{1}{(x-ia)^{n+1}}\right] \quad \text{(Using (3.4))} \end{aligned} \tag{3.23}$$

To make the result free from i, we put $x = r\cos\theta, a = r\sin\theta$ where $r = \sqrt{x^2+a^2}$ and $\theta = \tan^{-1}\dfrac{a}{x}$.

Therefore,

$$\begin{aligned}\frac{1}{(x+ia)^{n+1}} &= \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}} \\ &= \frac{1}{r^{n+1}}(\cos\theta + i\sin\theta)^{-(n+1)} \\ &= \frac{1}{r^{n+1}}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right] \qquad (3.24)\end{aligned}$$

$$\begin{aligned}\text{and } \frac{1}{(x-ia)^{n+1}} &= \frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} \\ &= \frac{1}{r^{n+1}}(\cos\theta - i\sin\theta)^{-(n+1)} \\ &= \frac{1}{r^{n+1}}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right] \qquad (3.25)\end{aligned}$$

Using (3.23)-(3.25), we have

$$\begin{aligned}y_n &= \frac{(-1)^n\, n!}{2}\left[\frac{1}{(x+ia)^{n+1}} + \frac{1}{(x-ia)^{n+1}}\right] \\ &= \frac{(-1)^n\, n!}{2r^{n+1}}\left[\cos(n+1)\theta - i\sin(n+1)\theta + \cos(n+1)\theta + i\sin(n+1)\theta\right] \\ &= \frac{(-1)^n\, n!}{2r^{n+1}} \cdot 2\cos(n+1)\theta \\ &= \frac{(-1)^n\, n!\cos(n+1)\theta}{r^{n+1}}\end{aligned}$$

In-text Exercise 3.3. Find the n$^{\text{th}}$ derivative of the following functions:

1. $y = \dfrac{x^2}{(x+2)(2x+3)}$
2. $y = \dfrac{1}{x^2+a^2}$

3.5 Leibnitz's Theorem

Leibnitz's theorem is used for finding the n$^{\text{th}}$ derivative of the product of two functions in the terms of successive derivatives of the functions.

Theorem 3.1 (Leibnitz's Theorem). *If u and v are functions of x having n successive derivatives, then for $n > 1$*

$$\begin{aligned}(uv)_n = &\binom{n}{0}u_n v + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 \cdots + \binom{n}{r}u_{n-r}v_r + \cdots \\ &+ \binom{n}{n-1}u_1 v_{n-1} + \binom{n}{n}uv_n\end{aligned}$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ for } r = 0, 1, 2, \ldots, n.$$

Example 3.16. Consider the function $y = x \ln x$. Let $u = x$ and $v = \ln x$. Therefore,

$$u_1 = 1, u_r = 0 \text{ for } r = 2, 3, \ldots, n$$
$$\text{and } v_r = \frac{(-1)^{r-1}(r-1)!}{x^r} \text{ for } r = 1, 2, \ldots, n. \tag{3.26}$$

Therefore, by using the Libnitz's Theorem, we have

$$\begin{aligned}
y_n = (uv)_n &= \binom{n}{0} u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 \cdots + \binom{n}{n-1} u_1 v_{n-1} + \binom{n}{n} u v_n \\
&= \binom{n}{n-1} u_1 v_{n-1} + \binom{n}{n} u v_n \quad (\text{ Using (3.26)}) \\
&= n \cdot 1 \cdot \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + 1 \cdot x \cdot \frac{(-1)^{n-1}(n-1)!}{x^n} \\
&= \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} [n - (n-1)] \\
&= \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}
\end{aligned}$$

Note. We can also solve this example by choosing $u = \ln x$ and $v = x$.

Example 3.17. Consider the function $y = x^2 \cos 3x$. Let $u = \cos 3x$ and $v = x^2$. Therefore,

$$u_r = 3^r \cos\left(\frac{r\pi}{2} + 3x\right) \text{ for } r = 1, 2, \ldots, n.$$
$$v_1 = 2x, v_2 = 2, v_r = 0 \text{ for } r = 3, 4, \ldots n \tag{3.27}$$

Therefore, by Libnitz's Theorem, we have

$$\begin{aligned}
y_n = (uv)_n &= \binom{n}{0} u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 \cdots + \binom{n}{n-1} u_1 v_{n-1} + \binom{n}{n} u v_n \\
&= \binom{n}{0} u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 \quad (\text{ Using (3.27)}) \\
&= 1 \cdot 3^n \cos\left(\frac{n\pi}{2} + 3x\right) \cdot x^2 + n \cdot 3^{n-1} \cos\left(\frac{(n-1)\pi}{2} + 3x\right) \cdot 2x \\
&\quad + \frac{n(n-1)}{2} \cdot 3^{n-2} \cos\left(\frac{(n-2)\pi}{2} + 3x\right) \cdot 2 \\
&= 3^{n-2} \left[9x^2 \cos\left(\frac{n\pi}{2} + 3x\right) + 6nx \cos\left(\frac{(n-1)\pi}{2} + 3x\right) \right. \\
&\quad \left. + n(n-1) \cos\left(\frac{(n-2)\pi}{2} + 3x\right)\right]
\end{aligned}$$

Note. We can also solve this example by choosing $u = x^2$ and $v = \cos 3x$.

In-text Exercise 3.4. Find the n$^{\text{th}}$ derivative of following functions:

1. $y = x^2 e^{3x}$

2. $y = x^3 \cos x$

Example 3.18. By using the Leibnitz's Theorem, prove that

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0,$$

where $y = a\cos(\ln x) + b\sin(\ln x)$.

Solution. Differentiating y with respect to x, we get

$$\begin{aligned} y_1 &= -a\sin(\ln x)\cdot\frac{1}{x} + b\cos(\ln x)\frac{1}{x} \\ \implies \quad x y_1 &= -a\sin(\ln x) + b\cos(\ln x) \end{aligned}$$

Differentiating both sides with respect to x, we have

$$\begin{aligned} x y_2 + y_1 &= -a\cos(\ln x)\cdot\frac{1}{x} - b\sin(\ln x)\cdot\frac{1}{x} \\ \implies \quad x^2 y_2 + x y_1 &= -a\cos(\ln x) - b\sin(\ln x) \\ \implies \quad x^2 y_2 + x y_1 + y &= 0 \end{aligned}$$

We now differentiate the above equation n times. To differentiating the product terms $x^2 y_2$ and $x y_1$, we use Leibnitz's Theorem. Note that $y_{n+2} = y_{2+n}$ is the n$^{\text{th}}$ derivative of y_2. We obtain

$$\begin{aligned} &\left[\binom{n}{0} y_{n+2}\cdot x^2 + \binom{n}{1} y_{n+1}\cdot 2x + \binom{n}{2} y_n \cdot 2\right] \\ &\qquad + \left[\binom{n}{0} y_{n+1}\cdot x + \binom{n}{1} y_n \cdot 1\right] + y_n = 0 \\ \implies \quad & x^2 y_{n+2} + 2nx y_{n+1} + n(n-1) y_n + x y_{n+1} + n y_n + y_n = 0 \\ \implies \quad & x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0 \end{aligned}$$

Example 3.19. For the function $y = \sin^{-1} x$, prove that

$$(1-x)^2 y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0.$$

Solution. Differentiating $y = \sin^{-1} x$ with respect to x we have,

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{1-x^2}} \\ \implies \quad y_1^2 &= \frac{1}{1-x^2} \\ \implies \quad (1-x^2) y_1^2 &= 1 \end{aligned}$$

Again differentiating the above equation with respect to x, we get

$$\begin{aligned} & -2x y_1^2 + (1-x^2)\cdot 2 y_1 y_2 = 0 \\ \implies \quad & (1-x^2) y_2 - x y_1 = 0 \quad \text{(Dividing both sides by } 2y_1) \end{aligned}$$

Now, differentiating n times by using the Leibnitz's Theorem, we get

$$\left[\binom{n}{0}y_{n+2}\cdot(1-x^2)+\binom{n}{1}y_{n+1}\cdot(-2x)+\binom{n}{2}y_n\cdot(-2)\right] - \left[\binom{n}{0}y_{n+1}\cdot x+\binom{n}{1}y_n\cdot 1\right]=0$$

$$\implies \left[(1-x^2)\cdot y_{n+2}+n\cdot(-2x)\cdot y_{n+1}+\frac{n(n-1)}{2}\cdot(-2)\cdot y_n\right] - [x\cdot y_{n+1}+n\cdot 1\cdot y_n]=0$$

$$\implies (1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n=0$$

$$\implies (1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$$

In-text Exercise 3.5. Solve the following questions:

1. If $y=\tan^{-1}x$, show that

$$(1-x)^2y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0.$$

2. If $y=e^{m\sin^{-1}x}$, show that

$$(1-x)^2y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0.$$

3.6 Summary

In this lesson we have discussed the following points:

1. If the function $y=f(x)$ is differentiable successively, then its successive derivatives are denoted as:

First derivative:	$f'(x)$	or	$y'(x)$	or	y_1	or	$\frac{dy}{dx}$	or	Dy
Second derivative:	$f''(x)$	or	$y''(x)$	or	y_2	or	$\frac{d^2y}{dx^2}$	or	D^2y
Third derivative:	$f'''(x)$	or	$y'''(x)$	or	y_3	or	$\frac{d^3y}{dx^3}$	or	D^3y
$\vdots$									
$\mathbf{n^{th}}$ **derivative:**	$f^{(n)}(x)$	or	$y^{(n)}(x)$	or	y_n	or	$\frac{d^ny}{dx^n}$	or	D^ny

2. n^{th} derivative of some well known functions are as follows:

(i) $\dfrac{d^n}{dx^n}(ax+b)^m=\dfrac{m!}{(m-n)!}a^n(ax+b)^{m-n},\quad m\geq n, m>0.$

(ii) $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right)=\dfrac{(-1)^n\,n!\,a^n}{(ax+b)^{n+1}}$

(iii) $\frac{d^n}{dx^n}\ln(ax+b) = \frac{(-1)^{n-1}\,(n-1)!\,a^n}{(ax+b)^n}$

(iv) $\frac{d^n}{dx^n}a^{mx} = m^n\,(\ln a)^n\,a^{mx}$

(v) $\frac{d^n}{dx^n}e^{mx} = m^n e^{mx}$

(vi) $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

(vii) $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(viii) $\frac{d^n}{dx^n}e^{ax}\sin(bx+c) = r^n e^{ax}\sin(bx+c+n\theta)$, where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

(ix) $\frac{d^n}{dx^n}e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\theta)$, where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

3. n[th] derivative of a rational function can be calculated with the help of partial fractions and De Moivre's theorem. We can take help of following table to calculate partial fractions:

S.N.	Rational Fraction	Partial Fraction Form
1.	$\frac{ax+b}{(x-p)(x-q)}$	$\frac{A}{x-p} + \frac{B}{x-q}$
2.	$\frac{ax+b}{(x-p)^2}$	$\frac{A}{x-p} + \frac{B}{(x-p)^2}$
3.	$\frac{ax^2+bx+c}{(x-p)(x-q)(x-r)}$	$\frac{A}{x-p} + \frac{B}{x-q} + \frac{C}{x-r}$
4.	$\frac{ax^2+bx+c}{(x-p)^2(x-q)}$	$\frac{A}{x-p} + \frac{B}{(x-p)^2} + \frac{C}{x-q}$
5.	$\frac{ax^2+bx+c}{(x-p)(x^2+qx+r)}$	$\frac{A}{x-p} + \frac{Bx+C}{x^2+qx+r}$

where A, B and C are constants.

4. **Leibnitz's Theorem**: If u and v are functions of x having n successive derivatives, then

$$(uv)_n = \binom{n}{0}u_n v + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 \cdots + \binom{n}{r}u_{n-r}v_r + \cdots + \binom{n}{n-1}u_1 v_{n-1} + \binom{n}{n}uv_n$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ for } r = 0, 1, 2, \dots, n.$$

3.7 Self-Assessment Exercises

1. Find the n$^{\text{th}}$ derivative of

(i) $y = \sin^3 x$

(ii) $y = \sin x \sin 2x \sin 3x$

(iii) $y = \sin^2 x \cos x$

(iv) $y = e^x \cos x \sin x$

(v) $y = e^x \sin^2 x$

(vi) $y = \dfrac{x}{(x-1)(2x+7)}$

(vii) $y = \dfrac{x^2+1}{(x-1)(x^2-4)}$

(viii) $y = \tan^{-1} x$

2. Show that

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{(-1)^n\, n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right].$$

3. If $y = \ln\left(x + \sqrt{1+x^2}\right)$, show that

(i) $(1+x^2)y_2 + xy_1 = 0$,

(ii) $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$.

4. If $y = e^{m\cos^{-1} x}$, show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$$

5. If $y = (\sin^{-1} x)^2$, show that

(i) $(1-x)^2 y_2 - xy_1 - 2 = 0$,

(ii) $(1-x)^2 y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$.

3.8 Solutions to In-text Exercises

Exercise 3.1

1. $12\left[(1-18x^6)\cos(2x^3)-18x^3\sin(2x^3)\right]$

Exercise 3.2

1. $5^n\,n!$
2. $\dfrac{(-1)^{n+2}\,(n+1)!\,2^n}{(2x+3)^{n+2}}$
3. $2^n e^{2x+3}$
4. $2^{n-1}\cos\left(\dfrac{n\pi}{2}+2x\right)$
5. $10^{n/2}\sin(3x+n\tan^{-1}3)$

Exercise 3.3

1. $\dfrac{(-1)^n\,n!}{2}\left[\dfrac{9\cdot 2^n}{(2x+3)^{n+1}}-\dfrac{8}{(x+2)^{n+1}}\right]$
2. $\dfrac{(-1)^n\,n!\,\sin(n+1)\theta}{ar^{n+1}}$, where $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\left(\frac{b}{a}\right)$.

Exercise 3.4

1. $3^{n-2}e^{3x}\left[9x^2+6nx+n(n-1)\right]$
2. $x^3\cos\left(x+\dfrac{n\pi}{2}\right)+3nx^3\cos\left(x+\dfrac{(n-1)\pi}{2}\right)+3n(n-1)x\cos\left(x+\dfrac{(n-2)\pi}{2}\right)+$
   $n(n-1)(n-2)\cos\left(x+\dfrac{(n-3)\pi}{2}\right)$

3.9 Suggested Readings

1. Narayan, S. & Mittal, P. K.(2019). Differential Calculus. S. Chand Publishing.
2. Anton, H., Bivens, I. C., & Davis, S. (2015). Calculus: Early Transcendentals. John Wiley & Sons.
3. Singh, J.P. (2017). Calculus , 2nd Edition, Ane Books Pvt Ltd.

Lesson - 4

Partial Differentiation

Structure

4.1 Learning Objectives

The learning objectives of this lesson are to:

- understand the concept of partial differentiation.
- learn to determine the partial derivatives of various functions.
- learn the geometric interpretation of partial derivatives.
- use the Euler's Theorem on homogeneous functions to solve various problems.

4.2 Introduction

In this lesson, we will extend the concept of differentiation of function of a single variable to function of several variables. If a function of more than one variable is differentiated with respect to one independent variable while keeping other variables as constants, the derivative we obtain is known as a partial derivative. Partial derivatives have application in finding maxima or minima of functions of several variables. For the limiting scope of this book, we will discuss mainly function of two variables. We will also discuss the Euler's Theorem on homogeneous function that displays relation between the dependent variable, independent variable, the partial derivatives of the dependent variable and the order of the homogeneous function in consideration.

4.3 Function of Two Variables

We are already familiar with a function of a single variable. We now define functions of two variables. However, the definition can be extended to the functions of more than two variables.

Definition 4.1. Let D be a subset of $\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}\}$. A real valued function f on D is a rule that assigns a unique real number $z = f(x, y)$ to each element (x, y) in D. Here

(i) D is called the domain of the function f.

(ii) The set $\{f(x, y) : (x, y) \in D\}$is called the range set of f.

(iii) z is the dependent variable and x and y are the independent variables.

Note. The graph of a function of two variables is called a **surface.**

Example 4.1. Following are some illustrations of functions of two variables:

1. $z = f(x, y) = x^2 + y^2$ with domain $D = \{(x, y) : -3 \leq x \leq 3, -4 \leq y \leq 4\}$.

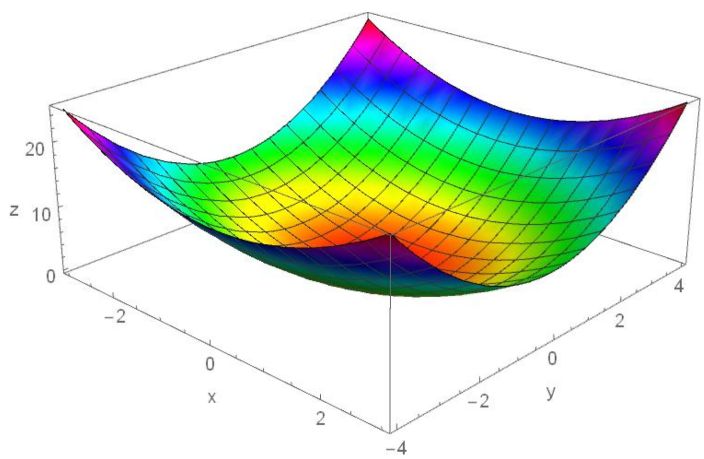


Figure 4.1: Graph of $f(x, y) = x^2 + y^2$.

2. $z = f(x, y) = 2\sin x + \sin y$ with domain $D = \{(x, y) : -2\pi \leq x, y \leq 2\pi\}$.

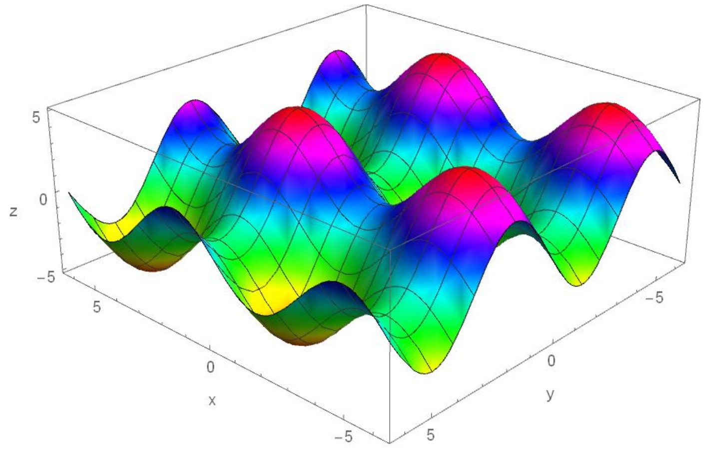


Figure 4.2: Graph of $f(x, y) = 2\sin x + \sin y$.

4.4 Partial Derivatives

4.4.1 Partial Derivatives of Function of Two Variables

Consider $z = f(x, y)$ be a function of two variables x and y. If we treat y as a constant, then $z = f(x, y)$ can be considered as a function of x alone and we can talk about the derivative of $z = f(x, y)$ with respect to x (keeping y as a constant). Similarly, we can also talk about the derivative of $z = f(x, y)$ with respect to y (keeping x as a constant)

Definition 4.2. Let $z = f(x, y)$ be a function of two variables x and y. We define

(i) The partial derivative of $z = f(x, y)$ with respect to x at the point $(x, y) = (a, b)$ as

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}, \tag{4.1}$$

provided the limit exists.

(ii) The partial derivative of $z = f(x, y)$ with respect to y at the point $(x, y) = (a, b)$ as

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b) = \lim_{k\to 0} \frac{f(a, b + k) - f(a, b)}{k}, \tag{4.2}$$

provided the limit exists.

Example 4.2. Consider the function

$$f(x, y) = 2x^2 + 5xy + y^2.$$

Find the partial derivatives $f_x(1, 3)$ and $f_y(1, 3)$.

Solution. Since

$$f(x,y) = 2x^2 + 5xy + y^2.$$

Therefore, we have

$$\begin{aligned} f_x(1,3) &= \lim_{h\to 0} \frac{f(1+h,3) - f(1,3)}{h} \\ &= \lim_{h\to 0} \frac{[2(1+h)^2 + 5\cdot(1+h)\cdot 3 + 3^2] - [2\cdot 1^2 + 5\cdot 1\cdot 3 + 3^2]}{h} \\ &= \lim_{h\to 0} \frac{2h^2 + 19h}{h} \\ &= \lim_{h\to 0}(2h + 19) \\ &= 19 \end{aligned}$$

Alternately, to obtain $f_x(1,3)$, we treat y as a constant in $f(x,y)$ and differentiate it with respect to x. Therefore,

$$\begin{aligned} & f_x(x,y) = 4x + 5y, \\ \implies \quad & f_x(1,3) = 4\cdot 1 + 5\cdot 3 = 19. \end{aligned}$$

Similarly,

$$\begin{aligned} f_y(1,3) &= \lim_{k\to 0} \frac{f(1,3+k) - f(1,3)}{k} \\ &= \lim_{h\to 0} \frac{[2\cdot 1^2 + 5\cdot 1\cdot(3+k) + (3+k)^2] - [2\cdot 1^2 + 5\cdot 1\cdot 3 + 3^2]}{k} \\ &= \lim_{k\to 0} \frac{k^2 + 11k}{k} \\ &= \lim_{k\to 0}(k + 11) \\ &= 11 \end{aligned}$$

Alternately, to obtain $f_y(1,3)$, we treat x as a constant in $f(x,y)$ and differentiate it with respect to y. Therefore,

$$\begin{aligned} & f_y(x,y) = 5x + 2y, \\ \implies \quad & f_y(1,3) = 5\cdot 1 + 2\cdot 3 = 11. \end{aligned}$$

Note. We note the following:

1. Equations (4.1) and (4.2) define the partial derivatives of f at the point (a,b). If the partial derivative of f with respect to x (or y) exists at all points of the domain, then we define

   (i) The partial derivative of $z = f(x,y)$ with respect to x as

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \tag{4.3}$$

(ii) The partial derivative of $z = f(x,y)$ with respect to y

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)} = f_y(x,y) = \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}, \tag{4.4}$$

where f, f_x and f_y have common domain.

2. The derivatives in (4.3) and (4.4) are called **first order** partial derivatives.

Example 4.3. Consider the function $f(x,y) = xe^{x^2+y^3} + y^2$. Therefore,

$$\begin{aligned} f_x(x,y) &= \left.\frac{d}{dx}f(x,y)\right|_{y=constant} \\ &= \left.\frac{d}{dx}\left[xe^{x^2+y^3}+y^2\right]\right|_{y=constant} \\ &= e^{x^2+y^3} + xe^{x^2+y^3}\cdot 2x + 0 \\ &= (1+2x^2)e^{x^2+y^3} \end{aligned}$$

and

$$\begin{aligned} f_y(x,y) &= \left.\frac{d}{dy}f(x,y)\right|_{x=constant} \\ &= \left.\frac{d}{dy}\left[xe^{x^2+y^3}+y^2\right]\right|_{x=constant} \\ &= xe^{x^2+y^3}\cdot 3y^2 + 2y \\ &= 3xy^2e^{x^2+y^3} + 2y \end{aligned}$$

In-text Exercise 4.1.

1. Calculate $f_x(x,y)$ and $f_y(x,y)$ for the following function:

   (i) $f(x,y) = (x^2+y^2)\sin(x^3-y^3)$

   (ii) $f(x,y) = \ln(x^2+y^2+x)$

2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function z given by the equation

$$yz - \ln x = x + y.$$

[Hint: Differentiate the given equation with respect to x, treating y as a constant to obtain $\frac{\partial z}{\partial x}$. Similarly, obtain $\frac{\partial z}{\partial y}$ by differentiating the given equation with respect to y by treating x as a constant.]

4.4.2 Geometric Interpretation of Partial Derivatives

We have studied earlier that for a function of single variable $y = f(x)$, the derivative of $f(x)$ at the point $x = a$ $\left(\text{namely} f^{'}(a)\right)$ is the slope of the tangent to curve of $y = f(x)$ at $x = a$. In a similar manner, we can also relate partial derivatives to slope of the tangents.

Let S be the surface representing the graph of $z = f(x, y)$ in Figure 4.3 and $P(a, b, c)$ be the point on the surface where $c = f(a, b)$. In this figure C_1 represents the curve $z = f(x, b)$, which is the intersection of the surface S with the vertical plane $y = b$. Then it represents a function of the single variable x. Let us denote it by $p(x)$. That is $p(x) = f(x, b)$. Therefore, $f_x(a, b) = p^{'}(a)$ is the slope of the tangent T_1 to the curve C_1 at the point P.

Similarly C_2 represents the curve $z = f(a, y)$, which is the intersection of the surface S with the horizontal plane $x = a$. Then it represents a function of the single variable y. Let us denote it by $q(y)$. That is $q(y) = f(a, y)$. Therefore, $f_y(a, b) = q^{'}(b)$ is the slope of the tangent T_2 to the curve C_2 at the point P.

In short,

1. $f_x(a, b) = \dfrac{\partial f}{\partial x}(a, b)$ represents the slope of the tangent line to the intersection of the graph of f with the plane $y = b$ at the point (a, b).

2. $f_y(a, b) = \dfrac{\partial f}{\partial y}(a, b)$ represents the slope of the tangent line to the intersection of the graph of f with the plane $x = a$ at the point (a, b).

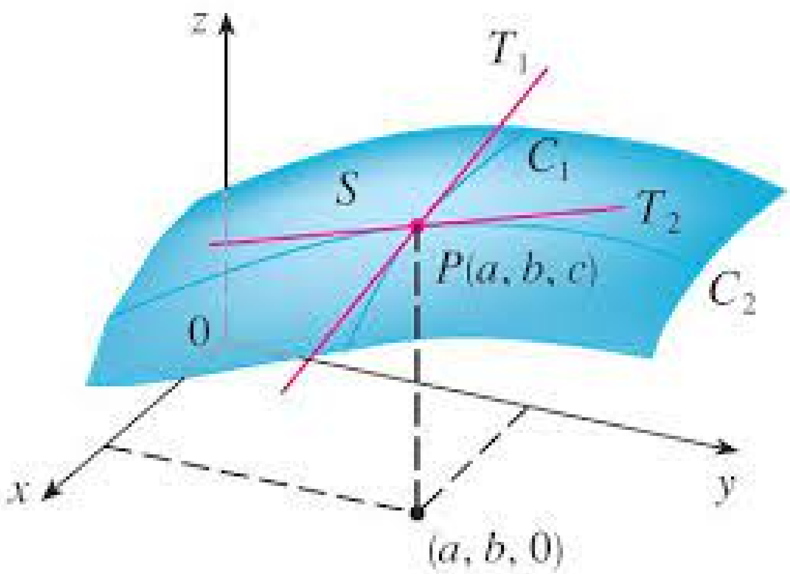


Figure 4.3: Intersection of planes and surface.

4.4.3 Partial Derivative of Function of Three Variables

Let us consider $z = f(x, y, t)$ be function of three variables x, y and t. Then the partial derivative of f with respect to x (or y or z) is calculated using differentiating the function

f with respect to x (or y or z) while treating the other two variables as constants. The three first order partial derivatives are denoted by

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial t}.$$

Example 4.4. Consider the function

$$f(x, y, t) = x^2y + xyt + xt^3 + yt + x^3 \sin t.$$

We have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x^2y + xyt + xt^3 + yt + x^3 \sin t\right), \\
&= 2xy + yt + t^3 + 3x^2 \sin t, \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2y + xyt + xt^3 + yt + x^3 \sin t\right), \\
&= x^2 + xt + t, \\
\text{and} \qquad \frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\left(x^2y + xyt + xt^3 + yt + x^3 \sin t\right), \\
&= xy + 3xt^2 + y + x^3 \cos t.
\end{aligned}$$

In-text Exercise 4.2. Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial t}$ for the following functions:

1. $f(x, y) = x^2 + y^2 + xyt + \ln xy$
2. $f(x, y) = e^{xt} + \sin(x^2y + y^2t + xt^2)$
3. $f(x, y) = \dfrac{xy^2 + 2t}{x + t}$

4.4.4 Partial Derivatives of Higher Order

For a function$z = f(x, y)$ of two variable x and y , the partial derivatives f_x and f_y may be constants or functions of x and y. So these functions can also have partial derivatives with respect to x and y. In this way, we obtain the **second order** partial derivatives of f with respect to x and y, defined as follows:

(i) $f_{xx} = (f_x)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$ (Differentiating with respect to x two times treating y as a constant)

(ii) $f_{yy} = (f_y)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$ (Differentiating with respect to y two times treating x as a constant)

(iii) $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$ (First differentiating with respect to x treating y as a constant, then with respect to y treating x as a constant)

(iv) $f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$ (First differentiating with respect to y treating x as a constant, then with respect to x treating y as a constant)

Theorem 4.1 (**Equality of Partial Derivatives**). *Let f be a function of two variables x and y having continuous second order partial derivatives f_{xy} and f_{yx}, then*

$$f_{xy} = f_{yx}.$$

Similarly, we can also define third order, forth order and higher order partial derivatives like

(i) $f_{xxx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial x^2}\right) = \dfrac{\partial^3 f}{\partial x^3}$,

(ii) $f_{yyx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial y^2}\right) = \dfrac{\partial^3 f}{\partial x \partial y^2}$,

(iii) $f_{xyx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y \partial x}\right) = \dfrac{\partial^3 f}{\partial x \partial y \partial x}$ and so on.

Example 4.5. Consider the function

$$f(x, y) = x^3y^3 + 2x^2y + 4xy^2 + 2x + 3y - 1.$$

We have

$$\frac{\partial f}{\partial x} = 3x^2y^3 + 4xy + 4y^2 + 2$$

$$\text{and} \qquad \frac{\partial f}{\partial y} = 3x^3y^2 + 2x^2 + 8xy + 3.$$

Therefore,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2y^3 + 4xy + 4y^2 + 2\right) = 6xy^3 + 4y,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2y^3 + 4xy + 4y^2 + 2\right) = 9x^2y^2 + 4x + 8y,$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3x^3y^2 + 2x^2 + 8xy + 3\right) = 6x^3y + 8x,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^3y^2 + 2x^2 + 8xy + 3\right) = 9x^2y^2 + 4x + 8y.$$

Here, we note that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

Example 4.6. Consider the function $f(x, y) = xe^{x^2+y^3} + y^2$. Since we have already calculated

$$f_x(x, y) = (1 + 2x^2)e^{x^2+y^3}$$

$$\text{and} \qquad f_y(x, y) = 3xy^2e^{x^2+y^3} + 2y.$$

Therefore,

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}\left((1+2x^2)e^{x^2+y^3}\right) \\ &= (1+2x^2)e^{x^2+y^3}\cdot 3y^2 \\ &= 3(1+2x^2)y^2e^{x^2+y^3} \end{aligned}$$

and

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}\left(3xy^2e^{x^2+y^3}+2y\right) \\ &= 3y^2e^{x^2+y^3}+3xy^2\cdot e^{x^2+y^3}\cdot 2x \\ &= 3(1+2x^2)y^2e^{x^2+y^3} \end{aligned}$$

Here also we note that $f_{xy} = f_{yx}$.

4.5 Homogeneous Functions

Definition 4.3 (**Homogeneous Function**)**.** A function f of two variables x and y is said to be a homogeneous function of degree (order) $r \in \mathbb{R}$, if

$$f(\alpha x, \alpha y) = \alpha^r f(x,y), \quad \alpha \neq 0 \tag{4.5}$$

or

$$f(x,y) = x^r g\left(\frac{y}{x}\right) \tag{4.6}$$

where g is a function of $\frac{y}{x}$.

Example 4.7. Consider the function

$$f(x,y) = x^5 + 3x^2y^3 + 6xy^4 - 4y^5.$$

Therefore, for $\alpha \neq 0$

$$\begin{aligned} f(\alpha x,\alpha y) &= (\alpha x)^5 + 3(\alpha x)^2(\alpha y)^3 + 6(\alpha x)(\alpha y)^4 - 4(\alpha y))^5 \\ &= \alpha^5\left(x^5+3x^2y^3+6xy^4-4y^5\right) \\ &= \alpha^5 f(x,y). \end{aligned}$$

Therefore, $f(x,y) = x^5 + 3x^2y^3 + 6xy^4 - 4y^5$ is a homogeneous function of degree 5.

Alternately,

$$\begin{aligned} f(x,y) &= x^5 + 3x^2y^3 + 6xy^4 - 4y^5 \\ &= x^5\left[1+3\frac{y^3}{x^3}+6\frac{y^4}{x^4}-4\frac{y^5}{x^5}\right] \\ &= x^5\left[1+3\left(\frac{y}{x}\right)^3+6\left(\frac{y}{x}\right)^4-4\left(\frac{y}{x}\right)^5\right] \\ &= x^5 g\left(\frac{y}{x}\right) \end{aligned}$$

where $g\left(\frac{y}{x}\right) = 1+3\left(\frac{y}{x}\right)^3+6\left(\frac{y}{x}\right)^4-4\left(\frac{y}{x}\right)^5$. Therefore, the given function is a homogeneous function of degree 5.

Example 4.8. Let $f(x,y) = \frac{x^2+y^2}{x^3-y^3}$. We can write

$$\begin{aligned} f(x,y) &= \frac{x^2+y^2}{x^3-y^3} \\ &= \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x^3\left(1-\frac{y^3}{x^3}\right)} \\ &= x^{-1}\frac{1+\left(\frac{y}{x}\right)^2}{1-\left(\frac{y}{x}\right)^3} \\ &= x^{-1}g\left(\frac{y}{x}\right) \end{aligned}$$

where $g\left(\frac{y}{x}\right) = \frac{1+\left(\frac{y}{x}\right)^2}{1-\left(\frac{y}{x}\right)^3}$. Hence, the given function is homogeneous of degree -1.

In-text Exercise 4.3. Verify that the following functions are homogeneous and find out the degree:

1. $f(x,y) = \frac{x^3+y^3}{x-y}$
2. $g(x,y) = \frac{\sqrt[3]{x^2-y^2}}{x^2+y^2}$

4.5.1 Euler's Theorem on Homogeneous Functions

Euler's Theorem displays a relation between the dependent variable, the independent variables, the partial derivatives of dependent variable with respect to the independent variables and the degree (order) of the homogeneous function.

Theorem 4.2 (Euler's Theorem). *If $z = f(x,y)$ is a homogeneous function of two variables x and y of degree r, then*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = rf(x,y). \tag{4.7}$$

Proof. Since $z = f(x,y)$ is a homogeneous function of degree r. Therefore,

$$f(x,y) = x^r g\left(\frac{y}{x}\right). \tag{4.8}$$

where g is a function of $\frac{y}{x}$. Therefore, we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= rx^{r-1}g\left(\frac{y}{x}\right) + x^r g'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)\\ &= rx^{r-1}g\left(\frac{y}{x}\right) - x^{r-2}yg'\left(\frac{y}{x}\right) \qquad (4.9)\\ \text{and}\quad \frac{\partial f}{\partial y} &= x^r g'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)\\ &= x^{r-1}g'\left(\frac{y}{x}\right) \qquad (4.10)\end{aligned}$$

Therefore, using (4.9) and (4.10), we have

$$\begin{aligned}x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\cdot\left[rx^{r-1}g\left(\frac{y}{x}\right) - x^{r-2}yg'\left(\frac{y}{x}\right)\right] + y\cdot x^{r-1}g'\left(\frac{y}{x}\right)\\ &= rx^r g\left(\frac{y}{x}\right) - x^{r-1}yg'\left(\frac{y}{x}\right) + x^{r-1}yg'\left(\frac{y}{x}\right)\\ &= rx^r g\left(\frac{y}{x}\right)\\ &= rf(x,y)\end{aligned}$$

□

Example 4.9. Verify that the function

$$f(x,y) = \frac{x^2+y^2}{x^3-y^3}$$

is a homogeneous function of x and y and it satisfies the Euler's Theorem.

Solution. We have

$$\begin{aligned}f(x,y) = \frac{x^2+y^2}{x^3-y^3} &= \frac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{x^3\left[1-\left(\frac{y}{x}\right)^3\right]}\\ &= \frac{1+\left(\frac{y}{x}\right)^2}{x\left[1-\left(\frac{y}{x}\right)^3\right]}\\ &= x^{-1}\frac{1+\left(\frac{y}{x}\right)^2}{1-\left(\frac{y}{x}\right)^3}\end{aligned}$$

Therefore, f is a homogeneous function of degree -1. Now we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x^3-y^3}\right)\\ &= \frac{(x^3-y^3)\cdot 2x-(x^2+y^2)\cdot 3x^2}{(x^3-y^3)^2}\\ &= \frac{2x^4-2xy^3-3x^4-3x^2y^2}{(x^3-y^3)^2}\\ &= \frac{-x^4-2xy^3-3x^2y^2}{(x^3-y^3)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x^3-y^3}\right)\\ &= \frac{(x^3-y^3)\cdot 2y-(x^2+y^2)\cdot(-3y^2)}{(x^3-y^3)^2}\\ &= \frac{2x^3y-2y^4+3x^2y^2+3y^4}{(x^3-y^3)^2}\\ &= \frac{2x^3y+3x^2y^2+y^4}{(x^3-y^3)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} &= x\left[\frac{-x^4-2xy^3-3x^2y^2}{(x^3-y^3)^2}\right]+y\left[\frac{2x^3y+3x^2y^2+2y^4}{(x^3-y^3)^2}\right]\\ &= \frac{-x^5-2x^2y^3-3x^3y^2+2x^3y^2+3x^2y^3+y^5}{(x^3-y^3)^2}\\ &= \frac{-x^5-x^3y^2+x^2y^3+y^5}{(x^3-y^3)^2}\\ &= \frac{x^2(y^3-x^3)+y^2(y^3-x^3)}{(x^3-y^3)^2}\\ &= \frac{(y^3-x^3)(x^2+y^2)}{(x^3-y^3)^2}\\ &= \frac{-(x^2+y^2)}{(x^3-y^3)}\\ &= (-1)f(x,y)\end{aligned}$$

Hence, the Euler's Theorem is satisfied.

In-text Exercise 4.4. Verify Euler's Theorem for the following functions:

1. $f(x,y) = x\ln\left(\dfrac{y}{x}\right)$
2. $f(x,y) = 9x^3+5x^2y+y^3$

Example 4.10. For $z = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin 2z.$$

Solution. We have

$$\begin{aligned} z &= \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right) \\ \implies \quad \tan z &= \frac{x^3+y^3}{x+y} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1+\frac{y}{x}\right]} = x^2 \cdot \frac{\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[1+\frac{y}{x}\right]}. \end{aligned} \tag{4.11}$$

Let $\tan z = u$, then

$$u = x^2 \cdot \frac{\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[1+\frac{y}{x}\right]}$$

is a homogeneous function of degree 2. Therefore, by the Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 \cdot u \tag{4.12}$$

Also,

$$\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = \sec^2 z\frac{\partial z}{\partial x} \\ \text{and} \quad &\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} = \sec^2 z\frac{\partial z}{\partial y} \end{aligned} \tag{4.13}$$

Therefore, from (4.12) and (4.13), we get

$$\begin{aligned} x \cdot \sec^2 z\frac{\partial z}{\partial x} + y \cdot \sec^2 z\frac{\partial z}{\partial y} &= 2\tan z \\ \implies \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= 2\frac{\tan z}{\sec^2 z} \\ &= 2\sin z\cos z \\ &= \sin 2z. \end{aligned}$$

Example 4.11. For $z = e^{\frac{x^2+y^2}{x+y}}$, prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z\ln z.$$

Solution. We have

$$
\begin{aligned}
& z = e^{\frac{x^2+y^2}{x+y}} \\
\implies \quad & \ln z = \frac{x^2+y^2}{x+y} = x \cdot \frac{\left[1+\left(\frac{y}{x}\right)^2\right]}{\left[1+\frac{y}{x}\right]}.
\end{aligned} \tag{4.14}
$$

Let $\ln z = u$, then

$$u = x \cdot \frac{\left[1+\left(\frac{y}{x}\right)^2\right]}{\left[1+\frac{y}{x}\right]}$$

is a homogeneous function of degree 1. Therefore, by the Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \cdot u \tag{4.15}$$

Also,

$$
\begin{aligned}
& \frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{1}{z}\frac{\partial z}{\partial x} \\
\text{and} \quad & \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{1}{z}\frac{\partial z}{\partial y}
\end{aligned} \tag{4.16}
$$

Therefore, from (4.15) and (4.16), we get

$$
\begin{aligned}
& x \cdot \frac{1}{z}\frac{\partial z}{\partial x} + y \cdot \frac{1}{z}\frac{\partial z}{\partial y} = \ln z \\
\implies \quad & x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \ln z
\end{aligned}
$$

In-text Exercise 4.5.

1. If $z = \sin^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2\tan z$.

2. If $z = \ln\left(\dfrac{x^5+y^5}{x^3+y^3}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2$.

4.6 Summary

In this lesson we have discussed the following points:

1. Let D be a subset of $\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R} = \{(x, y)|x, y \in \mathbb{R}\}\}$. A real valued function f on D is a rule that assigns a unique real number $z = f(x, y)$ to each element (x, y) in D. Here

   (i) D is called the domain of the function f.

(ii) The set $\{f(x,y) : (x,y) \in D\}$ is called the range set of f.

(iii) If $z = f(x,y)$, then z is the dependent variable and x and y are the independent variables.

2. The graph of a function of two variables represents a **surface.**

3. Let $z = f(x,y)$ be a function of two variables x and y. We define

(i) The partial derivative of $z = f(x,y)$ with respect to x at the point $(x,y) = (a,b)$ as

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}, \tag{4.17}$$

provided the limit exists.

(ii) The partial derivative of $z = f(x,y)$ with respect to y at the point $(x,y) = (a,b)$ as

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b) = \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}, \tag{4.18}$$

provided the limit exists.

4. **Geometric Interpretation of Partial Derivative:**

(i) $f_x(a,b) = \dfrac{\partial f}{\partial x}(a,b)$ represents the slope of the tangent line to the intersection of the graph of f with the plane $y = b$ at the point (a,b).

(ii) $f_y(a,b) = \dfrac{\partial f}{\partial y}(a,b)$ represents the slope of the tangent line to the intersection of the graph of f with the plane $x = a$ at the point (a,b).

5. The **second order** partial derivatives of f with respect to x and y are defined as follows:

(i) $f_{xx} = (f_x)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$ (Differentiating with respect to x two times treating y as a constant)

(ii) $f_{yy} = (f_y)_y = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$ (Differentiating with respect to y two times treating x as a constant)

(iii) $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$ (First differentiating with respect to x treating y as a constant, then with respect to y treating x as a constant)

(iv) $f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$ (First differentiating with respect to y treating x as a constant, then with respect to x treating y as a constant)

6. **Equality of second order partial derivatives:** Let f be a function of two variables x and y having continuous second order partial derivatives f_{xy} and f_{yx}, then

$$f_{xy} = f_{yx}.$$

7. **Homogeneous function:** A function f of two variables x and y is said to be a homogeneous function of degree (order) $r, r \in \mathbb{R}$ if

$$f(\alpha x, \alpha y) = \alpha^r f(x, y), \quad \alpha \neq 0$$

or

$$f(x, y) = x^r g\left(\frac{y}{x}\right)$$

where g is a function of $\frac{y}{x}$.

8. **Euler's Theorem on homogeneous functions:** If $f(x, y)$ is a homogeneous function of two variables x and y of degree r, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = rf(x, y).$$

4.7 Self-Assessment Exercises

1. Find the partial derivatives f_x and f_y for the following functions:

(i) $f(x, y) = 2x^2y + y^3 - 3xy^2$

(ii) $f(x, y) = \sin(x^2 - y^2)\cos(x^2 + y^2)$

(iii) $f(x, y) = e^{e^x + xy^2}$

(iv) $f(x, y) = \ln(x^3 + 2x^2y + y^2)$

(v) $f(x, y) = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

(vi) $f(x, y) = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$

2. Find the partial derivatives f_{xx}, f_{xy}, f_{yx} and f_{yy} for the following functions:

(i) $f(x, y) = xy^2 + e^{x+y^2}\sin x$

(ii) $f(x, y) = \sin(x^2 + y^3)\cos(x + y)$

(iii) $f(x, y) = \dfrac{xy + x^2}{x^2 + xy^3}$

(iv) $f(x, y) = \ln\left(\dfrac{x^2 + y^2}{x + y}\right)$

(v) $f(x, y) = e^{\sin(x+2y+xy)}$

3. Verify the Euler's Theorem for the following functions:

(i) $z = \dfrac{1}{x^2 + y^2}$

(ii) $z = x^3 \ln\left(\dfrac{y}{x}\right)$

(iii) $z = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$

(iv) $z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

4. Using Euler's Theorem, show that if $f(x, y) = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 0.$

5. If $\cos z = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, then prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + \dfrac{\cos z}{2 \sin z} = 0.$

6. If $z = \cot^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, then prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + \dfrac{\sin 2z}{4} = 0.$

4.8 Solutions to In-text Exercises

Exercise 4.1

1. (i) $f_x(x, y) = 2x \sin(x^3 - y^3) + 3x^2(x^2 + y^2) \cos(x^3 - y^3),$
   $f_y(x, y) = 2y \sin(x^3 - y^3) - 3y^2(x^2 + y^2) \cos(x^3 - y^3).$

   (ii) $f_x(x, y) = \dfrac{2x + 1}{x^2 + y^2 + x}, \;\; f_y(x, y) = \dfrac{2y}{x^2 + y^2 + x}$

2. $\dfrac{\partial z}{\partial x} = \dfrac{x^2 + 1}{xy}, \;\; \dfrac{\partial z}{\partial x} = \dfrac{1 - z}{y}.$

Exercise 4.2

1. $f_x(x, y, t) = 2x + yt + \dfrac{1}{x}, \;\; f_y(x, y, t) = 2y + xt + \dfrac{1}{y}, \;\; f_t(x, y, t) = xy$

2. $f_x(x, y, t) = te^{xt} + (2xy + t^2) \cos(x^2y + y^2t + xt^2),$
   $f_y(x, y, t) = (x^2 + 2yt) \cos(x^2y + y^2t + xt^2),$
   $f_t(x, y, t) = xe^{xt} + (y^2 + 2xt) \cos(x^2y + y^2t + xt^2).$

3. $f_x(x, y, t) = \dfrac{t(y^2 - 2)}{(x + t)^2}, \;\; f_y(x, y, t) = \dfrac{2xy}{(x + t)^2}, \;\; f_t(x, y, t) = \dfrac{x(2 - y^2)}{(x + t)^2}.$

Exercise 4.3

1. Degree $= 2$

2. Degree $= \dfrac{-4}{3}$

4.9 Suggested Readings

1. Narayan, S. & Mittal, P. K.(2019). Differential Calculus. S. Chand Publishing.
2. Anton, H., Bivens, I. C., & Davis, S. (2015). Calculus: Early Transcendentals. John Wiley & Sons.

Unit-2: Mean Value Theorem

4.9.1 Unit Overview

This unit on Mean Value Theorem is in continuation to the Unit-1. The topics discussed in Unit-1, such as limits, continuity and differentiability help us to prove important Mean Value Theorems, which have wide range of applications almost in every field. The other topics discussed such as Indeterminate forms, extrema of a function and Taylor's series expansions add to the values of this unit. This unit is further divided into four lesson.

In Lesson 5, we have discussed the two mean theorems, namely Rolle's Theorem and Lagrange's Mean Value Theorem. Basic applications of these theorems to establish some important inequalities and to check the monotonic behavior of functions are discussed.

In Lesson 7, the concept of convergence of a sequence and series and the concept of extrema of a function are discussed with applications.

In Lesson 8, we introduce some indeterminate forms on limits. The study of these indeterminate forms help us to evaluate many limits which are otherwise difficult to be evaluated.

In Lesson 6, we discussed Cauchy's Mean Value Theorem and the Taylor's Theorem. Taylor's series expansion and Maclaurin's series expansions of some functions are also discussed in this lesson.

The topics discussed in the above lessons are supported by examples, in-text exercises and self-assessment exercises.

Lesson - 5

Mean Value theorem

Structure

5.1 Learning Objectives

The learning objectives of this lesson are to:

- learn Rolle's Theorem and its applications.
- learn Lagrange's Mean Value Theorem and its applications.
- understand the concept of monotonicity of functions.

5.2 Introduction

In the previous lessons, we have learnt about the concept of continuity and differentiability of functions. In this lesson, we will learn how these concepts can be used to establish some standard theorems named as Rolle's Theorem and Lagrange's Mean Value Theorem. Rolle's Theorem is a special case of Mean Value Theorem. We will learn about the geometrical interpretation of both these theorems. The Mean Value Theorem is one of the important theorem in calculus, as it lays the foundation to many important results. We look at some of its applications to check the monotonic behavior of functions and to establish some inequalities at the end of this lesson.

5.3 Rolle's Theorem

This theorem is named after Michel Rolle a French mathematician. It states that if any real valued differentiable function have equal values at two distinct points, then it must have at least one stationary point (a point at which the derivative of the function become zero) somewhere between them.

Mathematically Rolle's Theorem can be stated as follows:

Theorem 5.1 (**Rolle's Theorem**). *Let $f(x)$ be a function defined on the closed interval $[a, b]$, such that*

1. *$f(x)$ is continuous on the closed interval $[a, b]$,*
2. *$f(x)$ is differentiable on the open interval (a, b),*
3. *$f(x)$ has same value at $x = a$ and b i.e. $f(a) = f(b)$,*

then there exists at-least one point c in (a, b) such that $f'(c) = 0$.

Proof. We know that a continuous function on a closed interval is bounded and attains its bounds therein. Since the given function $f(x)$ is continuous on the closed interval $[a, b]$, therefore it is bounded on $[a, b]$. Let m and M denote the bounds of $f(x)$ on $[a, b]$

$$\text{i.e. } m \leq f(x) \leq M \quad \forall \ x \in [a, b]. \tag{5.1}$$

Since, f attains its bounds, so there exist points d, c in $[a, b]$, such that

$$f(d) = m \text{ and } f(c) = M. \tag{5.2}$$

From equation (5.1) and (5.2), we get

$$f(d) \leq f(x) \leq f(c) \quad \forall \quad x \in [a, b]. \tag{5.3}$$

Case I. Let $m = M$.

Then from equation (5.1), we obtain $f(x) = m \ \forall \ x \in [a,b]$. That is, $f(x)$ is a constant function. Therefore, $f'(x) = 0 \ \forall \ x \in [a,b]$. Thus, the theorem is true in this case.

Case II. Let $M \neq m$.

Since $f(a) = f(b)$ and $M \neq m$, so at least one of the number M and m is different from $f(a)$ and $f(b)$.
Suppose $M \neq f(a)$ and $M \neq f(b)$.
That is $f(c) \neq f(a)$ and $f(c) \neq f(b)$, then using equation (5.2) we get $c \neq a$ and $c \neq b$, where $c \in [a,b]$.
Thus $c \in (a,b)$ and so by given condition (ii), $f'(c)$ exists
i.e.

$$f'(c) = Lf'(c) = Rf'(c) \tag{5.4}$$

From (5.3),

$$f(x) - f(c) \leq 0 \ \forall \ x \in [a,b]. \tag{5.5}$$

Now, $Lf'(c) = lim_{x \to c-} \dfrac{f(x) - f(c)}{(x - c)} \leq 0$, using (5.5). (Notice that $x \to c- \Rightarrow x < c$, i.e., $x - c < 0$.)

$$\Rightarrow \qquad f'(c) \geq 0, \text{ using } (5.4). \tag{5.6}$$

Now, $Rf'(c) = lim_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \leq 0$, using (5.5). (Notice that $x \to c+ \Rightarrow x > c$, i.e., $x - c > 0$.)

$$\Rightarrow \qquad f'(c) \leq 0, \text{ using } (5.4). \tag{5.7}$$

Hence, from (5.6) and (5.7), we get $f'(c) = 0, c \in (a,b)$. This proves the theorem.

□

Remark. If any of the conditions of Rolle's Theorem is not satisfied, then the conclusion may not hold. This is illustrated in the following examples.

Example 5.1. Consider the function

$$f(x) = \begin{cases} 2x, & 0 \leq x < 1 \\ 3, & x = 1 \end{cases}$$

Here, $f(x)$ is not continuous at $x = 1$. Therefore, the condition (i) of the hypothesis of Rolle's Theorem is violated The conclusion also does not hold as $f'(x) \neq 0$ at any point $x \in (0,1)$.

Example 5.2. Consider the function

$$f(x) = |x| \quad x \in [-1,1].$$

Here $f(x)$ is not differentiable at $0 \in (-1,1)$. Therefore, the condition (ii) of the hypothesis of Rolle's Theorem is violated The conclusion also does not hold as $f'(x) \neq 0$ at any point $x \in (-1,1)$ excluding $x = 0$.

Example 5.3. Consider the function

$$f(x) = x, \quad x \in [1, 2].$$

Here, $f(1) = 1$ and $f(2) = 2$, which means $f(1) \neq f(2)$. Therefore, the condition (iii) of the hypothesis of Rolle's Theorem is violated The conclusion also does not hold as $f'(x) = 1 \neq 0$ at any point $x \in (1, 2)$.

Thus, Rolle's Theorem does not hold true if any one of its of the condition is excluded.

5.3.1 Algebraic interpretation of Rolle's Theorem

Algebraically, Rolle's Theorem implies that, *Between any two zeros of a function satisfying the conditions of the Rolle's Theorem, there exists at least one zero of its derivative*.

Let $y = f(x)$ is any function defined on closed interval $[a, b]$ and satisfying the conditions of the Rolle's Theorem on $[a, b]$ and $f(a) = f(b) = 0$. That is, a and b are zeroes of $f(x)$. Then from Rolle's Theorem we can conclude that there exist c belongs to open interval (a, b) such that $f'(c) = 0$ or c is the zero of the function $f'(x)$.

5.3.2 Geometrical interpretation of Rolle's Theorem

If $y = f(x)$ is any real valued function defined on $[a, b]$ such that

(i) It is continuous on $[a, b]$ (i.e. continuous curve can be drawn from $x = a$ to $x = b$.)

(ii) It is differentiable on (a, b) (i.e. unique tangent can be drawn at each point $x \in (a, b)$.)

(iii) $f(a) = f(b)$ (i.e. ordinates at the end points are equal),

then there exist at least one point $P(c, f(c)), a < c < b$ on the curve $y = f(x)$ such that tangent at the point P is parallel to x-axis.

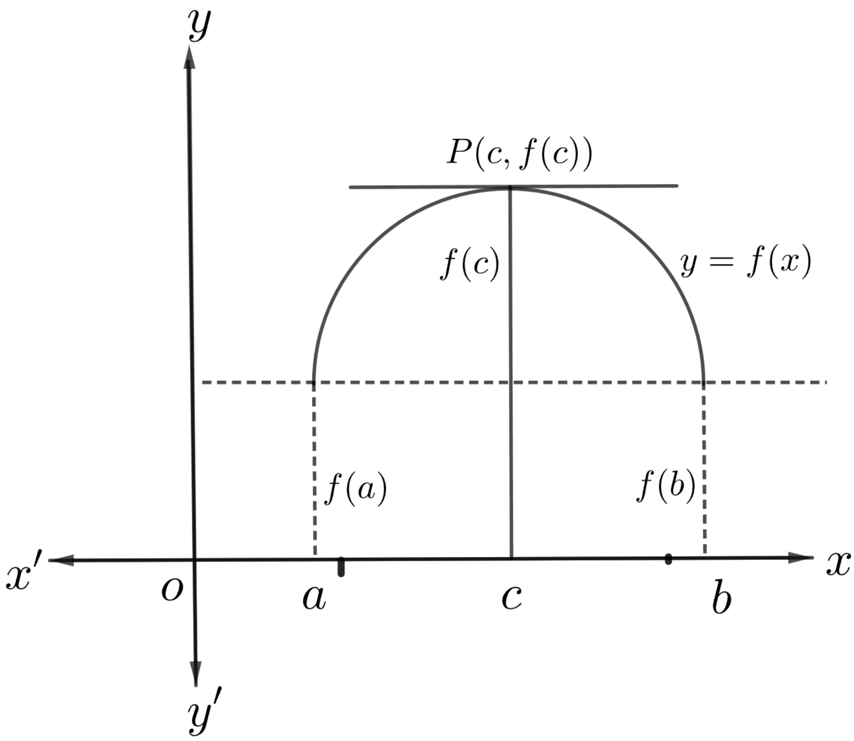


Figure 5.1: Geometrical interpretation of Rolle's Theorem.

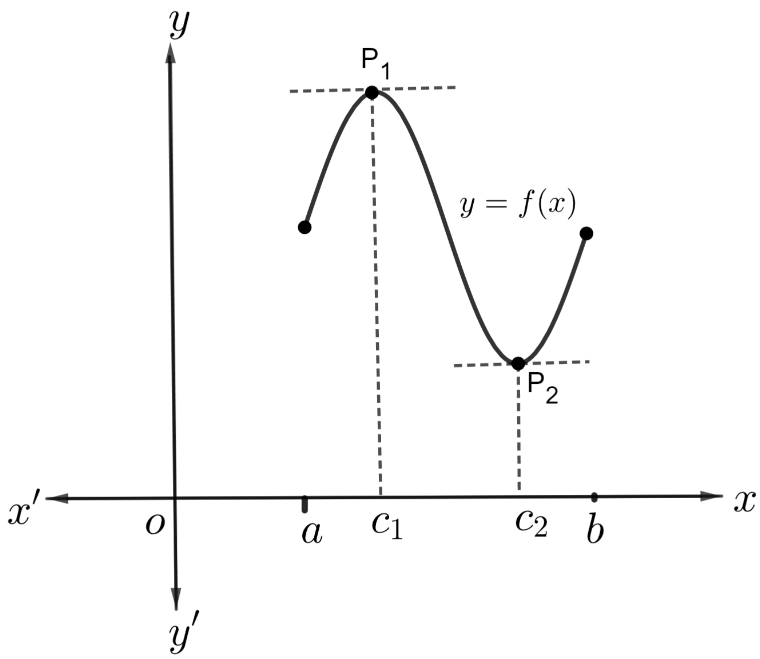


Figure 5.2: Geometrical interpretation of Rolle's Theorem

In the above figure 5.2 $f'(x) = 0$ at two points $x = c_1$ and $x = c_2$ while, the Rolle's

Theorem guarantees the existence of at least one such point.

Example 5.4. Verify Rolle's Theorem for $f(x) = \sqrt{1-x^2}$ on the interval $[-1, 1]$.

Solution. Given $f(x) = (1-x^2)^{\frac{1}{2}}, x \in [-1, 1]$. Therefore, $f(x)$ is a defined real function on $[-1, 1]$. To check the applicability of Rolle's Theorem we check the validity of all the three conditions of the hypothesis.

(i) Since $f(x)$ is an algebraic function in x and every algebraic function is continuous. Therefore, $f(x)$ is continuous in $[-1, 1]$.

(ii) Also,

$$f'(x) = \frac{1}{2}(1-x^2)^{\frac{1}{2}-1}. - 2x = \frac{-x}{\sqrt{1-x^2}}, \quad x \in (-1, 1).$$

$\Rightarrow f(x)$ is differentiable on $(-1, 1)$.

(iii) $f(-1) = 0 = f(1)$

Thus, $f(x)$ satisfies all the condition of Rolle's Theorem. Now to verify the conclusion, we have

$$f'(x) = \frac{-x}{\sqrt{1-x^2}}, \quad x \in (-1, 1)$$
$$\Rightarrow f'(x) = 0 \text{ at } x = 0.$$

Therefore, there exist a point $c = 0 \in (-1, 1)$ such that $f'(c) = 0$. Hence, the given function $f(x)$ satisfies all the conditions of the hypotheses as well as the conclusion of the Rolle's Theorem.

Example 5.5. Verify Rolle's Theorem for $f(x) = \log(x^2 + 2) - \log 3$ in $[-1, 1]$

Solution. Here,

$$f(x) = \log(x^2 + 2) - \log 3, \quad x \in [-1, 1]$$
$$\therefore f'(x) = \frac{2x}{x^2 + 2}, \quad x \in (-1, 1)$$

Therefore

(i) $f(x)$ is continuous on $[-1, 1]$, as $f(x)$ is the difference of the continuous function $\log(x^2 + 2)$ and $\log 3$.

(ii) $f(x)$ is derivable on $(-1, 1)$ and $f'(x) = \dfrac{2x}{x^2 + 2}$.

(iii) $f(-1) = \log(1 + 2) - \log 3 = 0 = f(1)$
$\therefore f(-1) = f(1)$.

Thus, all the conditions of Rolle's Theorem are satisfied. Hence, there exists at-least one point $c \in (-1, 1)$, such that $f'(c) = 0$. We have

$$f'(x) = \frac{2x}{x^2 + 2}$$
$$f'(c) = \frac{2c}{c^2 + 2} = 0 \text{ for } c = 0 \in (-1, 1).$$

Hence, Rolle's Theorem is verified.

Example 5.6. Show that between any two roots of the equation $e^x \cos x = 1$, there exists at least one root of the equation $e^x \sin x - 1 = 0$.

Solution. Let α and β be the two roots of $e^x \cos x = 1$. Let us define a function $f(x)$ as

$$f(x) = e^{-x} - \cos x \quad \text{for all } x \in [\alpha, \beta]. \tag{5.8}$$

Then

1. $f(x)$ is continuous on $[\alpha, \beta]$, as $\cos x$ and e^{-x} are continuous on $[\alpha, \beta]$.
2. $f(x)$ is differentiable on $[\alpha, \beta]$ and
3. $f(\alpha) = f(\beta) = 0$, from (5.8).

Also $f'(x) = -e^{-x} + \sin x$. Thus, all the conditions of Rolle's Theorem are satisfied by $f(x)$ on $[\alpha, \beta]$. Therefore, there exist at least one $c \in (\alpha, \beta)$ such that $f'(c) = 0$.

$$\text{i.e.,} \quad \sin c - e^{-c} = 0$$
$$\text{or} \quad e^c \sin c - 1 = 0$$

i.e. c is a root of the equation $e^x \sin x - 1 = 0$. Thus, there exist at least one root of the equation $e^x \sin x - 1 = 0$ in $[\alpha, \beta]$.

In-text Exercise 5.1. Solve the following questions:

1. Discuss the applicability of Rolle's Theorem for the following functions on the indicated intervals:

   (i) $f(x) = 2x^2 - 5x + 3$ on $[1, 3]$

   (ii)
   $$f(x) = \begin{cases} -4x + 5 & \text{if } 0 \le x \le 1 \\ 2x - 3 & \text{if } 1 < x \le 2 \end{cases}$$
   on $[0, 2]$.

   (iii) $f(x) = e^x(\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}.\right]$

   (iv) $f(x) = |x - 1|$ on $[-2, 2]$.

2. Using Rolle's Theorem, find a point on the curve $y = 16 - x^2$, $x \in [-1, 1]$, where the tangent is parallel to x- axis.

3. If the Rolle's Theorem holds for the function $f(x) = x^3 + bx^2 + cx, x \in [1, 2]$ at the point $\frac{4}{3}$. Find the values of b and c.

5.4 Lagrange's Mean Value Theorem

Lagrange's Mean Value Theorem (LMV Theorem) is a further extension of Rolle's Theorem. In this theorem the third condition that $f(a) = f(b)$ is removed. It concludes that there exists at least one point $c \in (a, b)$, such that the tangent line at $P(c, f(c))$ on the curve $y = f(x)$ is parallel to the secant line joining the points $A(a, f(a))$ and $B(b, f(b))$ on the curve. This theorem is also known as the **Fundamental Mean Value Theorem**. It is stated as following:

Theorem 5.2. *Let f be a function defined on the closed interval $[a, b], b > a$ such that it satisfies the following conditions:*

(i) $f(x)$ is continuous in closed interval $[a, b]$,

(ii) $f(x)$ is derivable in open interval (a, b).

Then, there exist at-least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

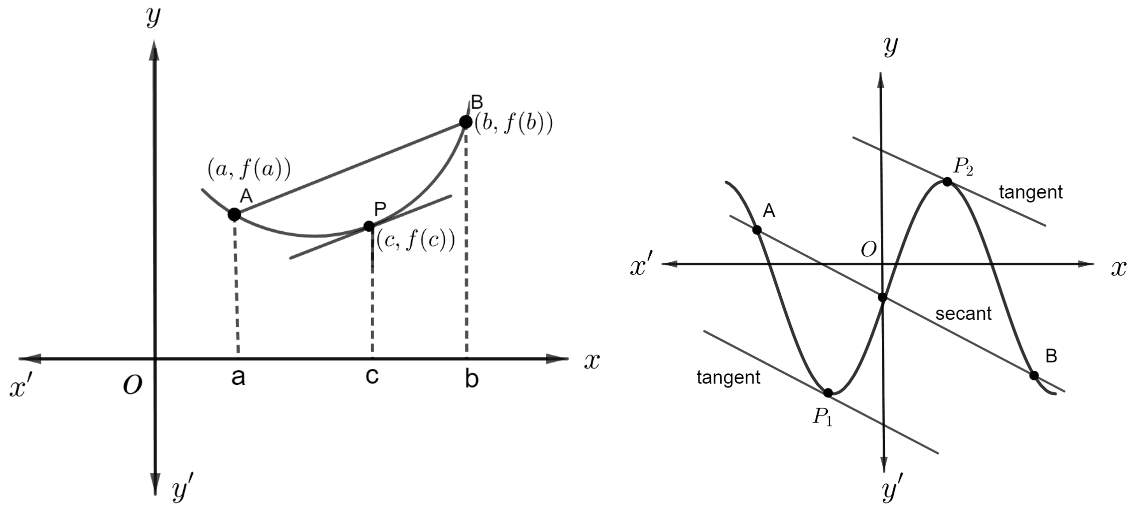


Figure 5.3: Graphical representation of Lagrange's Mean Value Theorem

Proof. Consider a function

$$\phi(x) = f(x) + Ax, \tag{5.9}$$

where A is a constant and we chose it in such a way that $\phi(a) = \phi(b)$. Now

$$\begin{aligned} \phi(a) &= \phi(b) \\ \implies \quad f(a) + Aa &= f(b) + Ab \\ \implies \quad f(b) - f(a) &= A(a - b) \\ \implies \quad A &= -\left\{\frac{f(b) - f(a)}{b - a}\right\}. \end{aligned} \tag{5.10}$$

Now, the function $\phi(x)$ in (5.9), where A is given by (5.10), satisfies the conditions:

1. $\phi(x)$ is continuous on $[a, b]$, as both $f(x)$ and Ax are continuous.
2. $\phi(x)$ is differentiable on $[a, b]$, as both $f(x)$ and Ax are differentiable on (a, b) and $\phi'(x) = f'(x) + A$.
3. $\phi(a) = \phi(b)$, by the choice of A.

Thus, all the conditions of Rolle's Theorem are satisfied by $\phi(x)$ on $[a, b]$. Hence, there exist at-least a point $c \in (a, b)$ such that $\phi'(c) = 0$. Now

$$\begin{aligned} & \phi(x) = f(x) + Ax \\ \implies & \phi'(x) = f'(x) + A \\ \implies & \phi'(c) = f'(c) + A = 0 \quad \implies A = -f'(c). \end{aligned} \tag{5.11}$$

From equation (5.10) and (5.11), we get

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This proves the theorem. □

5.4.1 Geometrical Interpretation of Lagrange's Mean Value Theorem

The conditions of Lagrange's Mean Value Theorem implies that

(i) $f(x)$ is continuous in the closed interval $[a, b]$ That is, the curve $y = f(x)$ is smooth from the point $A(a, f(a))$ to the point $B(b, f(b))$ and it has no break.

(ii) $f(x)$ is derivable in the open interval (a, b) That is, the tangent at each point of (a, b) is unique and non-vertical.

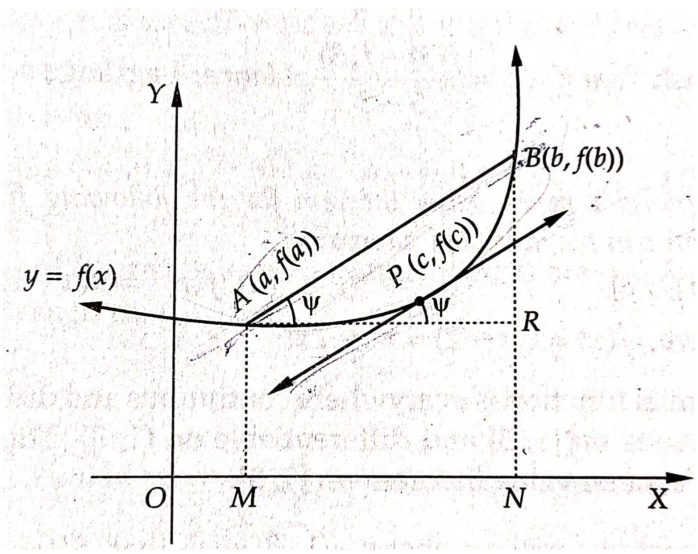


Figure 5.4: Geometrical interpretation of Lagrange's Mean Value Theorem

Let the end points $A(a, f(a))$ and $B(b, f(b))$ are joined by the chord AB and it makes an angle ψ with x-axis. Then from the triangle ARB the slope of the chord is

$$\tan\psi = \frac{BR}{AR} = \frac{f(b) - f(a)}{b - a}. \tag{5.12}$$

Also from Lagrange's Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad c \in (a, b). \tag{5.13}$$

Thus, from 5.12 and 5.13, we have

$$f'(c) = \tan\psi$$

slope of the tangent at $P(c, f(c))$ = Slope of the chord (secant) AB

Hence, in geometrical form Lagrange's Mean Value Theorem can be stated as 'if there is a continuous curve between the points A and B on the curve $y = f(x)$ having a unique tangent at each point between A and B, then there is at-least one point on the curve between A and B, where the tangent is parallel to the chord AB.

Note. 1. If in the hypothesis of Lagrange's Mean Value Theorem one more condition is added that is the value of the function at the end points are same i.e. $f(a) = f(b)$ Then by the conclusion of Lagrange's mean value theorem, there exist a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Since $f(b) = f(a)$, therefore $f'(c) = 0$ which is the conclusion of the Rolle's Theorem. Thus, Rolle's Theorem is a special case of Lagrange's Mean Value Theorem.

2. We also obtain from Lagrange's Mean Value Theorem that the average rate of change of a function on an interval is equal to the actual rate of change of the function at some point of the interval.

3. Lagrange's Mean Value Theorem may not hold if any one condition of the hypothesis is not satisfied. This is illustrated in the following example.

Example 5.7. Check the validity of the Lagrange's Mean Value Theorem for the following function $f(x),\ x \in [1, 2]$.

$$f(x) = \begin{cases} x^2 & \text{if } 1 < x < 2 \\ 2 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \end{cases}$$

Solution. Let, the Lagrange's Mean Value theorem be applicable for the given function. Then there exist at-least one point c in $(1, 2)$, such that

$$\frac{f(2) - f(1)}{2 - 1} = f'(c),$$

$$\implies \quad \frac{1-2}{2-1} = 2c$$

or

$$\implies \quad c = \frac{-1}{2} \notin (1,2).$$

Hence, the conclusion of the Lagrange's Mean Value theorem does not hold for the given function. It may be noticed that the given function is not continuous at $x = 1$ and $x = 2$ as

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} x^2 = 1 \neq f(1),$$
$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} x^2 = 4 \neq f(2)$$

Note. There may be some functions for which one or both the conditions of the hypothesis of Lagrange's theorem are not true but still a point $c \in (a,b)$ can be obtained for which the conclusion of the theorem holds true. In other words the conditions of the Lagrange's theorem are only sufficient but not necessary for the conclusion. This is illustrated in the next example.

Example 5.8. Consider a function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/4 \\ x & \text{if } 1/4 \leq x < 1/2 \\ (x/2)+1 & \text{if } 1/2 \leq x \leq 2 \end{cases}$$

Show that the function f is neither continuous in $[0,2]$ nor derivable in $(0,2)$, but at the point $x = 1/2$, the conclusion of the theorem holds.

Example 5.9. Verify Lagrange's Mean Value Theorem for $f(x) = \sin x$ in $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

Solution. Here $f(x) = \sin x$. Then $f(x)$ is a real function defined in $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$. Also,

(i) since, $\lim_{x\to a} f(x) = \lim_{x\to a} \sin x = \sin a = f(a) \ \forall \ a \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$. Therefore, $f(x)$ is continuous on $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

(ii) $f'(x) = \cos x$ for $x \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$. Therefore, $f(x)$ is derivable in $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$.

Thus, both the conditions of Lagrange's Mean Value Theorem are satisfied. Hence, there exists at-least one point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c), \quad \text{where } a = \frac{\pi}{2}, b = \frac{5\pi}{2} \tag{5.14}$$

We have,

$$\begin{aligned}
f(x) &= \sin x, \\
f(a) &= f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1, \\
f(b) &= f\left(\frac{5\pi}{2}\right) = \sin\frac{5\pi}{2} = 1, \\
\text{Also} \qquad f'(x) &= \cos x \\
\Longrightarrow \qquad f'(c) &= \cos c
\end{aligned}$$

Therefore, from (5.14), we get

$$\begin{aligned}
\frac{1-1}{\frac{5\pi}{2}-\frac{\pi}{2}} &= \cos c \\
\cos c &= 0 \\
c &= \frac{3\pi}{2} \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right).
\end{aligned}$$

Hence, Lagrange's Mean Value Theorem is verified.

Example 5.10. Find the point 'c' of the Lagrange's Mean Value Theorem if $f(x) = (x-1)(x-2)(x-3)$ and $a = 0, b = 4$.

Solution. Here $\quad f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$

(i) Since, $f(x)$ is a polynomial in x, therefore it is continuous in $[0, 4]$.

(ii) Also, $f'(x) = 3x^2 - 12x + 11$ which exists for all $x \in (0, 4)$. Thus, $f(x)$ is derivable in $(0, 4)$.

Since, both the condition of Lagrange's Mean Value Theorem are satisfied. Hence there must exist at-least one point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c), \text{ where } a = 0, b = 4 \tag{5.15}$$

Now,

$$\begin{aligned}
f'(x) &= 3x^2 – 12x + 11 \\
\Longrightarrow f'(c) &= 3c^2 – 12^c + 11 \\
f(b) &= f(4) = (4–1)(4-2)(4-3) = 3.2.1. = 6 \\
\text{and} \qquad f(a) &= f(0) = (-1)(-2)(-3) = -6.
\end{aligned}$$

Substituting all these values in (5.15), we get

$$\begin{aligned}
\frac{6-(-6)}{4-0} &= 3c^2 - 12c + 11 \\
3c^2 - 12c + 8 &= 0 \\
\text{i.e. } c &= \frac{12 \pm \sqrt{144-96}}{6} = \frac{6 \pm 2\sqrt{3}}{3}.
\end{aligned}$$

Both the above value of c lie in between $(0, 4)$. We note that, Lagrange's Mean Value Theorem guarantees the existence of at least one such point. Here, we have two points for which Lagrange's Mean Value Theorem is satisfied.

Example 5.11. Use Lagrange's Mean Value Theorem to determine a point on the curve $y = \sqrt{x^2 - 4}$ defined in $[2, 4]$, where the tangent is parallel to the chord joining the end points of the curve.

Solution. Given the function $y = f(x) = \sqrt{x^2 - 4}$, it is defined for $x \in [2, 4]$. Also,

1. $f(x)$ is continuous on $[2, 4]$.
2. $f'(x) = \dfrac{1}{2\sqrt{x^2 - 4}} \cdot 2x = \dfrac{x}{\sqrt{x^2 - 4}}$, which exists for all $x \in (2, 4)$. Therefore, $f(x)$ is derivable in $(2, 4)$.

Thus, both the conditions of Lagrange's Mean Value Theorem are satisfied. Hence, there exists at-least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c), \quad \text{where } a = 2, b = 4 \tag{5.16}$$

Now,

$$\begin{aligned} f'(x) &= \frac{x}{\sqrt{x^2 - 4}} \\ \implies \quad f'(c) &= \frac{c}{\sqrt{c^2 - 4}} \\ f(b) &= f(4) = \sqrt{4^2 - 4} = 2\sqrt{3} \\ \text{and} \qquad f(a) &= f(2) = \sqrt{2^2 - 4} = 0. \end{aligned}$$

Therefore, substituting all these values in (5.16), we get

$$\begin{aligned} \frac{2\sqrt{3} - 0}{4 - 2} &= \frac{c}{\sqrt{c^2 - 4}} \\ \implies \quad \frac{c}{\sqrt{c^2 - 4}} &= \sqrt{3} \\ \implies \quad 2c^2 &= 12 \\ \text{i.e.} \quad c &= \pm\sqrt{6}. \end{aligned}$$

Now, $c = +\sqrt{6} \in (2, 4)$. Also for, $x = \sqrt{6}$, $y = \sqrt{x^2 - 4} = \sqrt{6 - 4} = \sqrt{2}$. Thus, the tangent to the given curve at the point $(\sqrt{6}, \sqrt{2})$ is parallel to the chord joining the end points of the curve for $[2, 4]$.

Theorem 5.3. *If f satisfies all the conditions of Lagrange's Mean Value Theorem and if $f'(x) = 0 \ \forall \ x \in (a, b)$, then f is constant on $[a, b]$.*

Proof. Let x_1 and x_2 be any two points in $[a, b]$ such that $x_1 < x_2$. Let $f'(x) = 0\ \forall x \in (a, b)$. Then by the Lagrange's Mean value theorem, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0, \quad \text{for some} \quad c \in (x_1, x_2).$$

$\implies f(x_1) = f(x_2)\ \forall\ x_1, x_2 \in [a, b]$ and so f is constant on $[a, b]$. □

Alternative form of Lagrange's Mean Value Theorem

Let us take $h = b - a$. Then the interval $[a, b]$ becomes $[a, a + h]$. A point c in $[a, a + h]$ can be written in the form $a + \theta h$, where $0 < \theta < 1$. Hence Lagrange's theorem can be stated as follows:
Let f be a function defined on $[a, a + h]$, such that

(i) f is continuous in $[a, a + h]$

(ii) f is derivable in $[a, a + h]$

Then there exists at least one real number θ, $0 < \theta < 1$, such that :

$$f'(a + \theta h) = \frac{f(a + h) - f(a)}{h},$$

equivalently, $f(a + h) = f(a) + hf'(a + \theta h)$.

Example 5.12. Prove that for any quadratic function $px^2 + qx + r$, the value of θ in the Lagrange's Theorem is always $1/2$ irrespective of the values of p, q, r, a, h.

Solution. Let $f(x) = px^2 + qx + r$ and the interval is $[a, a + h]$. Since $f(x)$ is a polynomial function, therefore

(i) $f(x)$ is continuous on $[a, a + h]$

(ii) $f(x)$ is derivable on $(a, a + h)$

Therefore, by the Lagrange's Mean Value Theorem there exist at least one $\theta \in (0, 1)$, satisfying

$$f(a + h) = f(a) + hf'(a + \theta h)$$

i.e.

$$p(a + h)^2 + q(a + h) + r = pa^2 + qa + r + h(2p(a + \theta h) + q)$$
$$\Rightarrow p(a + h)^2 – a^2 + qh = 2aph + 2p\theta h^2 + gh$$
$$\Rightarrow ph(2a + h) = 2aph + 2p\theta h^2$$
$$\Rightarrow \theta = 1/2$$

Since, θ is independent of p, q, r, a, h. So, the value of θ is $1/2$, irrespective of the values of p, q, r, a and h.

In-text Exercise 5.2. Solve the following questions:

1. Examine the applicability of Lagrange's Mean Value Theorem for the following functions:

   (i)
   $$f(x) = \begin{cases} 2 + x^3 & \text{if } x \leq 1 \\ 3x & \text{if } x > 1 \end{cases}$$
   on $[-1, 2]$.

   (ii) $f(x) = \log x$ on $[1, e]$.

   (iii) $f(x) = (x-1)(x-2)(x-3)$ on $[0, 4]$.

   (iv) $f(x) = x^3 - 5x^2 - 3x$ on $[1, 3]$.

2. If a and b are distinct real numbers, show that there exist a real number c between a and b such that
   $$a^2 + ab + b^2 = 3c^2.$$

3. Show that Lagrange's Mean Value Theorem is not applicable to the function $f(x) = \frac{1}{x}$ on $[-1, 1]$.

4. Find a point on the parabola $y = (x-4)^2$, where the tangent is parallel to the chord joining $(4, 0)$ and $(5, 1)$.

5.5 Applications of Mean Value Theorem to monotonic functions and inequalities

In this section, we will study the application of Mean Value Theorem for finding the monotonic functions and establish inequalities using the concept of monotone functions.

5.5.1 Monotone functions

Definition 5.1. A function f defined on a interval $[a, b]$ is said to be ***monotonically increasing or simply increasing,*** if for x_1, x_2 in $[a, b]$

$$f(x_1) \leq f(x_2) \quad \text{whenever} \quad x_1 \leq x_2.$$

Definition 5.2. A function f defined on a interval $[a, b]$ is said to be ***strictly increasing,*** if for x_1, x_2 in $[a, b]$
$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2.$$

Definition 5.3. A function f defined on a interval $[a, b]$ is said to be ***monotonically decreasing or simply decreasing,*** if for x_1, x_2 in $[a, b]$

$$f(x_1) \geq f(x_2) \quad \text{whenever} \quad x_1 \leq x_2.$$

Definition 5.4. A function f defined on a interval $[a, b]$ is said to be ***strictly decreasing,*** if for x_1, x_2 in$[a, b]$
$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2.$$

Definition 5.5. A function f defined on a interval $[a, b]$ is said to be ***monotone or strictly monotone,*** if f is either an increasing (or strictly increasing) function or a decreasing (or strictly decreasing) function.

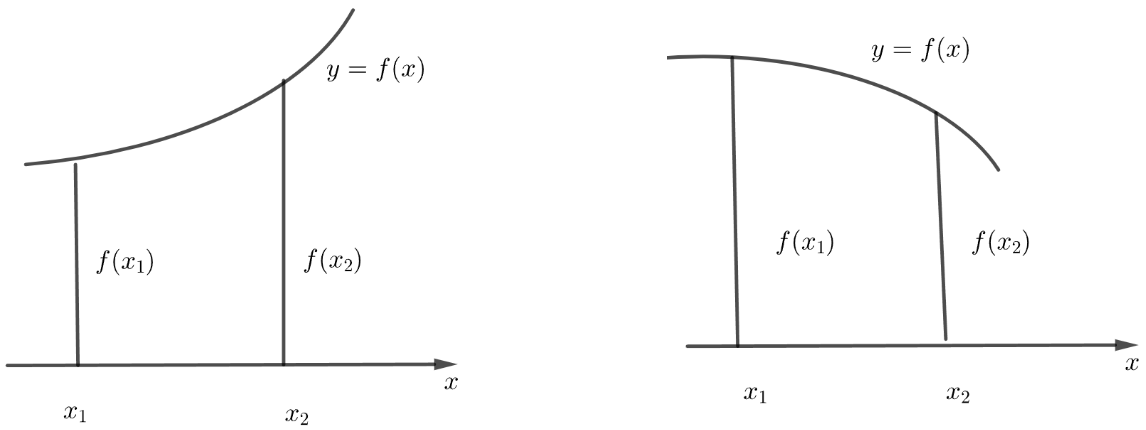


Figure 5.5: Graphs for monotonic functions: Figure (a) represent monotonically increasing function while Figure (b) represent monotonically decreasing function.

Theorem 5.4. ***(Necessary and sufficient condition)*** *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function on (a, b). Then*

1. *f is increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.*
2. *f is decreasing on (a, b) if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.*

Proof. (i) **Necessary condition**

Consider an arbitrary point $x_0 \in (a, b)$. If the function f is increasing on (a, b), then by definition, we can write;

$$\forall x \in (a, b) : x \geq x_0 \implies f(x) \geq f(x_0);$$
$$\forall x \in (a, b) : x \leq x_0 \implies f(x) \leq f(x_0).$$

By above result, we can write as

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0, \quad \text{where } \; x \neq x_0 \tag{5.17}$$

In the limit as $x \to x_0$, the left hand side of the inequality is equal to the derivative of the function at the point x_0, that is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \geq 0, \tag{5.18}$$

This relation is valid for any $x_0 \in (a, b)$.

Sufficient condition

Let x_1 and x_2 be any two points of $[a, b]$ such that $x_1 \leq x_2$. Let $f'(x) \geq 0 \ \forall x \in (a, b)$. Then by the Lagrange's Mean Value Theorem,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad \text{for some} \quad c \in (x_1, x_2).$$

$$\Rightarrow \qquad f(x_2) - f(x_1) = (x_2 - x_1)f'(c),$$

Since, $f'(c) \geq 0$ and $x_2 - x_1 \geq 0$, therefore $f(x_2) - f(x_1) \geq 0$. Hence, $f(x_2) \geq f(x_1)$ when $x_2 \geq x_1, \quad x_1, x_2 \in (a, b)$. Thus, f is increasing on (a, b).

(ii) By proceeding as in part (i), we can show that f is decreasing on (a, b) if and only if $f'(x) \leq 0$ for all $x \in (a, b)$. □

Example 5.13. Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x + 20$ is (i) increasing (ii) decreasing.

Solution. We have

$$f(x) = 2x^3 + 9x^2 + 12x + 20.$$

$$\therefore \quad f'(x) = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2).$$

(i) For $f(x)$ to be increasing, we must have $f'(x) \geq 0$

$\Rightarrow 6(x^2 + 3x + 2) \geq 0$
$\Rightarrow (x^2 - 3x + 2) \geq 0 \quad [\because \ 6 > 0 \ \text{and} \ 6(x^2 + 3x + 2) \geq 0 \therefore x^2 + 3x + 2 \geq 0]$
$\Rightarrow (x + 1)(x + 2) \geq 0$
$\Rightarrow x \leq -2 \ \text{or} \ x \geq -1$
$\Rightarrow x \in (-\infty, -2] \cup [-1, \infty)$

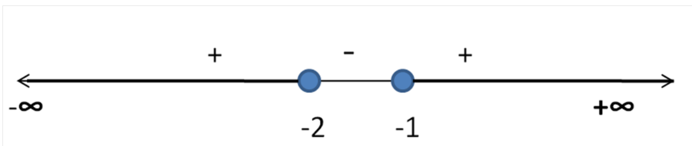


Figure 5.6: Signs of $f'(x)$ for different values of x.

So, $f(x)$ is increasing on $(-\infty, -2] \cup [-1, \infty)$.

(ii) For $f(x)$ to be decreasing, we must have $f'(x) \leq 0$

$\Rightarrow 6(x^2 + 3x + 2) \leq 0$
$\Rightarrow (x^2 + 3x + 2) \leq 0 \quad [\because \ 6 > 0 \ \text{and} \ 6(x^2 + 3x + 2) \leq 0 \therefore x^2 + 3x + 2 \leq 0]$
$\Rightarrow (x + 1)(x + 2) \leq 0$
$\Rightarrow -2 \leq x \leq -1$

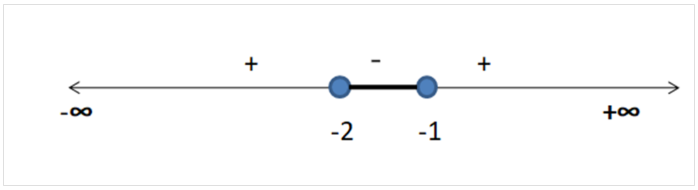


Figure 5.7: Signs of $f'(x)$ for different values of x.

So, $f(x)$ is decreasing on $[-2,-1]$.

Example 5.14. Find the intervals in which the function $f(x) = x^4 - \frac{x^3}{3}$ is increasing or decreasing.

Solution. We have

$$f(x) = x^4 - \frac{x^3}{3}$$

$$\therefore \quad f'(x) = 4x^3 - x^2 = x^2(4x-1)$$

(i) For $f(x)$ to be increasing, we must have $f'(x) \geq 0$

$$\Rightarrow x^2(4x-1) \geq 0$$
$$\Rightarrow (4x-1) \geq 0 \text{ and } x \neq 0$$
$$\Rightarrow 4x \geq 1 \text{ and } x \neq 0 \Rightarrow x \geq \frac{1}{4} \Rightarrow x \in \left[\frac{1}{4}, \infty\right).$$

So, $f(x)$ is increasing on $\left[\frac{1}{4}, \infty\right)$.

(ii) For $f(x)$ to be decreasing, we must have $f'(x) \leq 0$

$$\Rightarrow x^2(4x-1) \leq 0$$
$$\Rightarrow (4x-1) \leq 0 \text{ and } x \neq 0 \qquad [\because x^2 > 0]$$
$$\Rightarrow 4x \leq 1 \text{ and } x \neq 0 \Rightarrow x \leq \frac{1}{4} \text{ and } x \neq 0 \Rightarrow x \in (-\infty, 0) \cup \left(0, \frac{1}{4}\right].$$

So, $f(x)$ is decreasing on $(-\infty, 0) \cup \left(0, \frac{1}{4}\right]$.

Note. The above mentioned theorem 5.4 is stated regarding monotonic functions. Similar theorem, as stated below, holds for strictly monotonic functions.

Theorem 5.5. ***(Necessary and sufficient condition)*** *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function on (a, b). Then*

1. *f is strictly increasing on (a, b) if and only if $f'(x) > 0$ for all $x \in (a, b)$.*

2. *f is strictly decreasing on (a, b) if and only if $f'(x) < 0$ for all $x \in (a, b)$.*

Example 5.15. Find the intervals in which $f(x) = 2\log(x-2) - x^2 + 4x + 1$ is strictly increasing or strictly decreasing.

Solution. We have, $f(x)$ is well defined for all $x > 2$.
Now,

$$f(x) = 2\log(x-2) - x^2 + 4x + 1$$
$$\Rightarrow \quad f'(x) = \frac{2}{x-2} - 2x + 4 = \frac{-2x^2 + 8x - 6}{x-2} = \frac{-2(x-1)(x-3)}{x-2}.$$

For $f(x)$ to be increasing, we must have $f'(x) > 0$

$$\text{or} \quad \frac{-2(x-1)(x-3)}{x-2} > 0$$
$$\text{or} \quad \frac{(x-1)(x-3)}{x-2} < 0.$$

Since, $x > 2$ we have $x - 2 > 0$ and $x - 1 > 0$. Therefore, $\dfrac{(x-1)(x-3)}{x-2} < 0$ when $x - 3 < 0$ That is, when $x < 3$.
Thus, $f'(x) > 0$ when $x \in (2, 3)$. $\implies$ $f(x)$ is strictly increasing on $(2, 3)$.

For $f(x)$ to be decreasing, we must have $f'(x) < 0$

$$\text{or} \quad \frac{-2(x-1)(x-3)}{x-2} < 0$$
$$\text{or} \quad \frac{(x-1)(x-3)}{x-2} > 0$$

That is when $x-3 > 0$ or $x > 3$, as $x-1 > 0$ and $x-2 > 0$. So, $f(x)$ is strictly decreasing on $(3, \infty)$.

5.5.2 Inequalities

Here, we will establish some important inequalities by using Lagrange's Mean Value Theorem and also by using the concept of monotone functions.

Example 5.16. Use Mean Value Theorem to prove that

$$1 + x < e^x < 1 + xe^x \quad \forall x > 0$$

Solution. Consider $f(x) = e^x,\ x \in [0, x]$. Here, f is continuous on $[0, x]$ and derivable on $(0, x)$, therefore by the Mean Value Theorem there exists some $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$
$$\text{or} \quad \frac{e^x - e^0}{x - 0} = e^c$$
$$\text{or} \quad \frac{e^x - 1}{x} = e^c. \tag{5.19}$$

$$\begin{aligned}
\text{Now} \quad & 0 < c < x \\
\implies \quad & e^0 < e^c < e^x, \quad \text{as } e^x \text{ is an increasing function on} (0, \infty). \\
\text{or} \quad & 1 < e^c < e^x \\
\text{or} \quad & 1 < \frac{e^x - 1}{x} < e^x \qquad \text{[Using 5.19]} \\
\text{or} \quad & x < e^x - 1 < x \cdot e^x \\
\implies \quad & 1 + x < e^x < 1 + xe^x, \qquad \forall x > 0.
\end{aligned}$$

Example 5.17. Using Lagrange's Mean Value Theorem, show that

$$\frac{x}{1+x} < \log_e(1+x) < x, \quad x > 0$$

Solution. Let $f(x) = \log_e(1+x),\ x > 0$

$$\implies f'(x) = \frac{1}{1+x}.$$

Then, f is continuous in $[0, x]$ and derivable in $(0, x)$. Therefore, by Lagrange's Mean Value Theorem, there exists $\theta \in (0, 1)$, such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\theta x)$$

or

$$\log_e(1+x) = \frac{x}{1+\theta x} \qquad [\because f(0) = \log_e 1 = 0] \tag{5.20}$$

Now $0 < \theta < 1$ and $x > 0 \Rightarrow \theta x < x$

$$\Rightarrow 1 + \theta x < 1 + x \Rightarrow \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$\Rightarrow \frac{x}{1+\theta x} > \frac{x}{1+x} \Rightarrow \frac{x}{1+x} < \frac{x}{1+\theta x} \tag{5.21}$$

Again $0 < \theta < 1$ and $x > 0 \Rightarrow 1 < 1 + \theta x$

$$\Rightarrow \frac{1}{1+\theta x} < 1 \Rightarrow \frac{x}{1+\theta x} < x \tag{5.22}$$

From (5.21) and (5.22), we obtain

$$\frac{x}{1+x} < \frac{x}{1+\theta x} < x \tag{5.23}$$

Now, from (5.20) and (5.23), we obtain

$$\frac{x}{1+x} < \log_e(1+x) < x.$$

Example 5.18. Prove that $\tan x > x$ whenever $0 < x < \frac{\pi}{2}$.

Solution. Let c be any real number such that $0 < c < \frac{\pi}{2}$. Let us consider the function

$$f(x) = \tan x - x \ \ \forall \ \ x \in [0, c]$$

Then, f is continuous on $[0, c]$ as well as derivable on $(0, c)$. Now,

$$f'(x) = \sec^2 x - 1 = \tan^2 x > 0 \ \text{ for } \ 0 < x < c.$$

Thus, f is strictly increasing in $[0, c]$

$$\Rightarrow \quad f(c) > f(0) \ \text{ for } \ c > 0.$$

But $f(0) = 0$. Therefore $f(c) > 0 \Rightarrow \tan c - c > 0$. Since c is any real number such that $0 < c < \frac{\pi}{2}$, therefore

$$\tan x - x > 0, \text{ or } \tan x > x \ \text{ whenever } \ 0 < x < \frac{\pi}{2}.$$

Example 5.19. Show that, for all $x > 0$

$$e^x > 1 + x$$

Solution. We define the function $f(x)$ as

$$f(x) = e^x - (1 + x), \ \ x > 0.$$

Then $f(x)$ is a differentiable function for $x > 0$. Let us define another function $g(x)$ as

$$g(x) = f'(x) = e^x - 1 \ \ \forall \ \ x > 0.$$

$$\Longrightarrow \ g'(x) = e^x > 0 \text{ for all } x > 0.$$

$\Rightarrow g$ is a strictly increasing function for $x > 0$. Therefore

$$x > 0 \Rightarrow g(x) > g(0)$$

$$\text{i.e.} \quad e^x - 1 > e^0 - 1$$

$$\text{or} \quad e^x - 1 > 0$$

$$\Rightarrow f'(x) > 0 \ \ \forall \ \ x > 0.$$

$\Rightarrow f$ is an increasing function of x.

$$\therefore x > 0 \Rightarrow f(x) > f(0)$$

$$i.e. \quad e^x - (1 + x) > e^0 - (1 + 0) = 0$$

$$\Rightarrow e^x > (1 + x) \ \ \forall \quad x > 0.$$

In-text Exercise 5.3. Solve the following questions:

1. Separate the interval in which $f(x) = x^3 + 8x^2 + 5x - 2$ is increasing or decreasing.

2. Use Lagrange's Mean Value Theorem to show that

$$(b-a)\sec^2 a < \tan b - \tan a < (b-a)\sec^2 b,$$

where $0 < a < b < \frac{\pi}{2}$.

3. Using Lagrange's Mean Value Theorem, prove that

$$e^x > 1 + x + x^2.$$

4. Show that $x(\sin x)^{-1}$ increases for $0 < x < \frac{\pi}{2}$.

5. Find the interval in which $f(x) = \dfrac{x}{\log x}$ is increasing or decreasing.

6. Find the value of k for which $f(x) = kx^3 - 9kx^2 + 9x + 3$ is increasing on $\mathbb{R}$.

5.6 Summary

We have discussed following topics in this lesson:

1. Rolle's Theorem is a particular case of Lagrange's Mean Value Theorem.

2. Lagrange's Mean Value Theorem: If f be a function defined in the closed interval $[a, b]$ such that it is continuous in closed interval $[a, b]$, derivable in open interval (a, b), then there exist at-least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3. We obtain from the Lagrange's Mean Value Theorem that the average rate change in the value of the function in an interval is equal to the actual rate of change of the function at some point in the interval.

4. Applications of the Mean Value Theorem.

5. Monotone functions and their applications to establish some inequalities.

5.7 Self-Assessment Exercises

1. Let $f(x) = x^{\frac{2}{3}}$, $a = -1, b = 8$. Show that there is no real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

2. If $f : [-5, 5] \to \mathbb{R}$ is differentiable and if $f'(x)$ doesn't vanish anywhere, then prove that $f(-5) \neq f(5)$.

3. Discuss the applicability of Rolle's Theorem for the following functions on the indicated intervals:

   (i) $f(x) = x(x-4)^2$ on $[0, 4]$.

   (ii) $f(x) = \sin^4 x + \cos^4 x$ on $\left[0, \frac{\pi}{2}.\right]$

   (iii) $f(x) = [x]$ for $-1 \leq x \leq 1$, where $[x]$ denotes the greatest integer not exceeding x.

4. Verify that on the curve $f(x) = ax^2 + bx + c$, the chord joining the points $(p, f(p))$ and $(q, f(q))$ is parallel to the tangent at the point $x = \dfrac{p+q}{2}$.

5. Discuss the validity of the Rolle's Theorem for

$$f(x) = (x-c)^m (x-d)^n$$

   in $[a, b]$; where m, n being positive integers.

6. Use Rolle's Theorem to show that the equation

$$x^3 + 4x - 1 = 0,$$

   has exactly one real root.

7. Verify Lagrange's Mean Value Theorem for the following functions:

   (i) $f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$.

   (ii) $f(x) = |x|$ on $[-5, 5]$.

   (iii) $f(x) = \sqrt{x^2 - 4}$ on $[2, 4]$.

   (iv) $f(x) = 1 - (x-1)^{\frac{2}{3}}$ on $[0, 2]$.

8. Using Lagrange's Mean Value Theorem show that

   (i) $x - \dfrac{x^3}{6} < \sin x < x$.

   (ii) $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

9. Let f and g be differentiable function on $[0, 1]$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$ and $g(1) = 2$. Show that there exist $c \in (0, 1)$ such that $f'(c) = 2g'(c)$.

10. Show that the function $3x^3 - 9x^2 + 9x + 7$ is strictly increasing in every interval.

11. Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x + 20$ is (i) increasing (ii) decreasing.

12. Find the interval in which the given function

$$f(x) = (x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$$

   is increasing and decreasing.

13. Using Lagrange's Mean Value Theorem show that

$$\frac{x}{1+x^2} < \tan^{-1} x < x \text{ for } x > 0.$$

14. Find the intervals in which the function f given by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}, \quad 0 \le x \le 2\pi.$$

is increasing and decreasing.

15. Prove that the function

$$f(x) = x^3 - 3x^2 + 3x - 100$$

is increasing on $\mathbb{R}$.

16. Establish the Jordan's Inequality

$$1 < \frac{x}{\sin x} \le \frac{\pi}{2} \text{ for } 0 < x \le \frac{\pi}{2}.$$

17. Establish the Bernoulli's inequality

$$(1+x)^p \ge 1 + px \text{ for } x > -1 \text{ and } p > 1.$$

5.8 Solutions to In-text Exercises

Exercise 5.1

1. (i) Not applicable
   (ii) Not applicable
   (iii) Rolle's Theorem is applicable at the point is $c = \pi$.
   (iv) Not applicable
2. $(0, 16)$
3. $b = -5$, $c = 8$.

Exercise 5.2

1. (i) Since the function $f(x)$ is continuous and differential at $[-1, 2]$, hence the Mean Value Theorem is applicable. The value of c is obtained as $c = \pm\frac{\sqrt{5}}{3}$.
   (ii) The Mean Value Theorem is applicable and the value of c is obtained as $c = e - 1$.
   (iii) The Mean Value Theorem is applicable and the value of c is obtained as $c = 3$.
   (iv) The Mean Value Theorem is applicable and the value of c is obtained as $c = \frac{7}{3}$.

2. Applying Lagrange's Mean Value Theorem to $f(x) = x^3$ in $[a, b]$, the required result can be obtained.

3. As the function is not continuous and differentiable at $[-1, 1]$.

4. $\left(\frac{9}{2}, \frac{1}{4}\right)$.

Exercise 5.3

1. The given function is increasing on $(-\infty, -5]$, $\left[-\frac{1}{3}, \infty\right)$ and decreasing on $\left[-5, -\frac{1}{3}\right]$.

5. The given function is increasing on (e, ∞), and decreasing on $(0, e) - \{1\}$.

6. $f(x)$ is increasing on $\mathbb{R}$, if $k \in \left(0, \frac{1}{3}\right)$.

5.9 Suggested Readings

1. Narayan, Shanti (Revised by Mittal, P. K.). Differential Calculus. S. Chand, Delhi, 2019.

2. Prasad, Gorakh (2016). Differential Calculus (19th ed.) Pothishala Pvt. Ltd. Allahabad.

3. Thomas Jr., George B., Weir, Maurice D.,Hass, Joel (2014). Thomas Calculus (13th ed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 6

Taylor's Series and Maclaurin's Series

Structure

6.1 Learning Objectives

The learning objectives of this lesson are to:

- study the Cauchy's Mean Value Theorem and its applications.
- explain the meaning and significance of Taylor's theorem.

- learn to obtain the Taylor series expansion of a function.
- obtain the Maclaurin's series expansions of some standard functions.

6.2 Introduction

We have already learnt the applications of Rolle's theorem and Lagrange's Mean Value Theorem in the previous lesson 5. Taylor's theorem, which we will study in this lesson, can be regarded as a general form of Lagrange's Mean Value Theorem when the function is differentiable successively n times, $n > 1$. In this lesson, we examine how functions may be expressed in terms of power series. This is an extremely useful way of expressing a function since we can replace complicated functions in terms of simple polynomials.

6.3 Cauchy's Mean Value Theorem

Cauchy's Mean Value Theorem is a generalized form of the Lagrange's Mean Value Theorem. This theorem is also called the "**Second Mean Value Theorem**". It establishes the relationship between the derivative of the two functions and the change in these functions on a finite interval.

Theorem 6.1 (Cauchy's Mean Value Theorem). *Let f and g be two functions defined on the closed interval* $[a, b]$*, such that*

(i) $f(x)$ *and* $g(x)$ *both are continuous on* $[a, b]$.

(ii) $f(x)$ *and* $g(x)$ *both are differentiable on* (a, b).

(iii) $g'(x) \neq 0 \quad \forall x \in (a, b)$.

Then, there exist a point $c \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. We assume that $g(a) \neq g(b)$. If $g(a) = g(b)$, then $g(x)$ satisfies all the conditions of Rolle's theorem. Hence, there exist $c \in (a, b)$ such that $g'(c) = 0$ which contradict the condition (iii) of the theorem. Thus, $g(a) \neq g(b)$. We define a function $\phi(x) = f(x) + Ag(x)$, where, A is constant to be determined. Assume $\phi(a) = \phi(b)$, then

$$f(a) + Ag(a) = f(b) + Ag(b)$$
$$A = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{6.1}$$

Also, ϕ satisfies all the conditions of Rolle's theorem, i.e.

(i) Since f and g both are continuous on $[a, b]$, hence $\phi(x)$ is continuous.

(ii) f and g are derivable on (a, b), therefore $\phi(x)$ is derivable on (a, b).

(iii) $\phi(a) = \phi(b)$.

Thus, there exist $c \in (a, b)$ such that $\phi'(c) = 0$ i.e.

$$\begin{aligned} f'(c) + Ag'(c) &= 0 \\ \implies -A &= \frac{f'(c)}{g'(c)} \end{aligned} \tag{6.2}$$

Therefore from, 6.1 and 6.2, we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{6.3}$$

□

Remark. In the Cauchy's Mean Value Theorem, if we take $g(x) = x$, then $g'(x) = 1$. Also, $g(b) = b$ and $g(a) = a$. Thus, from 6.3, we get

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

which is the result of the Lagrange's Mean Value Theorem. Hence the Lagrange's Mean Value Theorem is a particular form of the Cauchy's Mean Value Theorem.

Example 6.1. Verify the Cauchy's Mean Value Theorem for $f(x) = x^2, \; g(x) = x^3$ in $[1, 2]$.

Solution. Here $f(x) = x^2$ and $g(x) = x^3$

(i) f and g being polynomial functions, they are continuous on $[1, 2]$.

(ii) f and g being polynomial functions, they are derivable on $(1, 2)$.

(iii) $g'(x) = 3x^2 \neq 0$ for all $x \in (1, 2)$.

Thus, the conditions of Cauchy's Mean Value Theorem are satisfied. Therefore, there exists some point $c \in (1, 2)$ such that

$$\begin{aligned} \frac{f(2) - f(1)}{g(2) - g(1)} &= \frac{f'(c)}{g'(c)} \\ \implies \frac{4 - 1}{8 - 1} &= \frac{2c}{3c^2} \\ \implies \frac{3}{7} &= \frac{2}{3c} \\ \implies c &= \frac{14}{9}, \quad \text{which lies in } \; (1, 2). \end{aligned}$$

Hence, Cauchy's Mean Value Theorem is verified.

Example 6.2. Show that

$$\frac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} = \cot\theta, \;\; 0 < \alpha < \beta < \frac{\pi}{2}$$

Solution. Consider two functions $f(x)$ and $g(x)$:

$$f(x) = \sin x, \quad g(x) = \cos x \;\; \text{for all} \;\; x \in [\alpha, \beta], \quad 0 < \alpha < \beta < \frac{\pi}{2}.$$

We apply Cauchy's Mean Value Theorem on $f(x)$ and $g(x)$ on the interval $[\alpha, \beta]$. We have

(i) $f(x) = \sin x$ and $g(x) = \cos x$ are continuous functions in the closed interval $[\alpha, \beta]$.

(ii) Since, $f'(x) = \cos x$ and $g'(x) = -\sin x$. Therefore, both the functions f and g are derivable in (α, β).

(iii) $g'(x) = -\sin x \neq 0$ for all $x \in (\alpha, \beta)$.

Thus, all the conditions of Cauchy's Mean Value Theorem are satisfied and so there exists some point $\theta \in (\alpha, \beta)$, such that

$$\begin{aligned}
& \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)} \\
\Rightarrow \quad & \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta} \\
\Rightarrow \quad & \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = -\cot\theta \\
\Rightarrow \quad & \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \;\; \text{where} \;\; 0 < \alpha < \beta < \frac{\pi}{2}.
\end{aligned}$$

In-text Exercise 6.1. Solve the following questions:

1. Verify Cauchy's Mean Value Theorem for

   (i) $f(x) = \sin x, g(x) = \cos x$ in $[-\pi/2, 0]$.

   (ii) $f(x) = e^x, g(x) = e^{-x}$ in $[0, 1]$.

2. Let the function ϕ be continuous in $[a, b]$ and derivable in (a, b). Show that there exists a point c in (a, b) such that

$$2c\,(\phi(a) - \phi(b)) = \phi'(c)(a^2 - b^2).$$

6.4 Taylor's Theorem

In the lesson 5, we have discussed Mean Value Theorems, which use the first order derivatives of functions. We generalize the concept to the functions those are differentiable k times(say), successively and obtain Taylor's theorem. Taylor's theorem gives an approximation of a k-times differentiable function around a given point by a polynomial of degree k, called the kth-order Taylor polynomial.

Theorem 6.2 (Taylor's Theorem with Lagrange's form of Remainder). *If a function f is defined on $[a, a+h]$, such that*

(i) f *and the derivatives* f', f'',..., $f^{(n-1)}$ *are continuous on* $[a, a+h]$,

(ii) the n^{th} *derivative* $f^{(n)}$ *exist on* $(a, a+h)$,

then there exists at least one point θ, $0 < \theta < 1$, *such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h).$$

Proof. Consider the function

$$\begin{aligned} F(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + ... \\ ... + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{(a+h-x)^n}{(n)!}A, \end{aligned} \tag{6.4}$$

where A is a constant to be chosen so that $F(a) = F(a+h)$.

Now from equation (6.4), we have

$$F(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}A, \tag{6.5}$$

$$\text{and} \quad F(a+h) = f(a+h). \tag{6.6}$$

Therefore $F(a) = F(a+h)$ gives

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}A, \tag{6.7}$$

which gives the value of A. With the choice of A, the function $F(x)$ satisfies all the conditions of Rolle's Theorem on $[a, a+h]$, as

1. $F(x)$ is continuous on $[a, a+h]$. As $f, f', f'' \ldots, f^{(n-1)}$ are continuous on $[a, a+h]$.

2. $F(x)$ is differentiable on $(a, a+h)$. As $f, f', f'' \ldots, f^{(n-1)}$ are differentiable on $(a, a+h)$.

3. $F(a) = F(a+h)$

Hence, from Rolle's Theorem there exist at-least one real number θ, $0 < \theta < 1$ such that

$$F'(a+\theta h) = 0. \tag{6.8}$$

Differentiating equation (6.4), w.r.t x, we get

$$\begin{aligned}
F'(x) &= f'(x) \\
&+ (a+h-x)f''(x) - f'(x) \\
&+ \frac{(a+h-x)^2}{2!}f'''(x) + \frac{2(a+h-x)}{2!}(-1)f''(x) \\
&+ \dots\dots\dots \\
&+ \dots\dots\dots \\
&+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - \frac{(n-1)(a+h-x)^{n-2}}{(n-1)!}f^{(n-1)}(x) \\
&+ \frac{n(a+h-x)^{n-1}(-1)A}{n!} \\
\text{or}\quad F'(x) &= \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - \frac{(a+h-x)^{n-1}}{(n-1)!}A \\
&= \frac{(a+h-x)^{n-1}}{(n-1)!}[f^{(n)}(x) - A]
\end{aligned}$$

For $x = a + \theta h$, we get

$$F'(a+\theta h) = \frac{[h(1-\theta)]^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A]. \tag{6.9}$$

From (6.8) and (6.9), we get

$$f''(a+\theta h) - A = 0 \implies A = f^n(a+\theta h).$$

Therefore, substituting the above value of A in (6.7), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h),$$

which is the required result.

□

Remark. Following are some important remarks:

1. The $(n+1)th$ term i.e. $\frac{h^n}{n!}f^{(n)}(a+\theta h)$ is called the Lagrange's remainder after n terms and is denoted by $\mathbf{R_n}$. Thus, the above theorem is called **Taylor's Theorem with Lagrange's Form of Remainder**

2. Putting $n = 1$ in Taylor's theorem, we get $f(a+h) = f(a) + hf'(a+\theta h)$ where $0 < \theta < 1$, which is the conclusion of Lagrange's Mean Value Theorem. Hence we conclude that Mean Value Theorem is a particular case of Taylor's Theorem.

3. If the remainder $\mathbf{R_n}$ is expressed as

$$\mathbf{R_n} = \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h), \tag{6.10}$$

then the above theorem is called **Taylor's Theorem with Cauchy's Form of Remainder** and $\mathbf{R_n}$ is called Cauchy's Remainder after n terms.

4. **Alternate form of Taylor's Theorem**
   If we choose $b = a + h$ then, $h = b - a$ and $c = a + \theta h = a + \theta(b - a)$ and $c \in (a, b)$. Therefore, alternately Taylor's Theorem can be concluded as follows

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(c), \quad [a < c < b].$$

Example 6.3. By using Taylor's Theorem with Lagrange's form of remainder, show that Show that

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \ldots + (-1)^{n-1}\frac{h^n}{n(x+\theta h)^n}, \; 0 < \theta < 1, h > 0.$$

Solution. Let $f(x+h) = \log(x+h)$. Therefore,

$$\begin{aligned}
f(x) &= \log x, \quad (\log x \text{ is natural logarithm}) \\
f'(x) &= \frac{1}{x} \\
f''(x) &= -\frac{1}{x^2} \\
f'''(x) &= \frac{(-1)(-2)}{x^3} = (-1)^2\frac{2!}{x^3} \\
&\cdots \\
&\cdots \\
f^{(n)}(x) &= (-1)^{n-1}\frac{(n-1)!}{x^n} \\
\therefore \quad f^{(n)}(x+\theta h) &= (-1)^{n-1}\frac{(n-1)!}{(x+\theta h)^n}.
\end{aligned}$$

Then, by using Taylor's theorem with Lagrange's form of remainder, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots \frac{h^2 n - 1}{(n-1)!}f^{n-1}(x) + \frac{h^n}{n!}f^n(x+\theta h).$$

$$\begin{aligned}
\therefore \quad \log(x+h) &= \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3 2!}{3! x^3} + \ldots \frac{h^{n-1}}{(n-1)!}(-1)^{n-2}\frac{(n-2)!}{x^{n-1}} + \frac{h^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(x+\theta h)^n} \\
&= \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} + \ldots (-1)^{n-2}\frac{h^{n-1}}{(n-1)x^{n-1}} + (-1)^{n-1}\frac{h^n}{n(x+\theta h)^n}.
\end{aligned}$$

Hence, the result.

6.4.1 Taylor's Infinite Series

If a function $f(x)$ possesses continuous derivatives of all orders in $(a, a+h)$, then for every integer n, howsoever large, there corresponds a Taylor's development with Lagrange's form of remainder, viz

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n$$

where,

$$R_n = \frac{h^n}{n!}f^n(a+\theta h), 0 < \theta < 1. \tag{6.11}$$

It may be written as $f(a+h) = S_n + R_n$,

where

$$S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a). \tag{6.12}$$

Now if R_n converges to 0 as $n \to \infty$, then

$$\lim_{n\to\infty} S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$$

and therefore, we can write

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots.$$

This is known as Taylor's infinite series expansion of $f(a+h)$. It is stated as follows:

Theorem 6.3 (Taylor's infinite series expansion). *If a function $f(x)$ defined on $[a, a+h]$ is such that*

(i) $f(x)$ possesses continuous derivatives of all orders in $(a, a+h)$.

(ii) For $0 < \theta < 1$, Taylor's remainder $R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h)$ tends to 0 as $n \to \infty$,

then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \ldots. \tag{6.13}$$

Other Forms of Taylor's Infinite Series :

1. Replacing a by x , in equation (6.13), we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots \tag{6.14}$$

2. Putting $a+h=b$ or $h=b-a$, in (6.13), we have

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\ldots+\frac{(b-a)^n}{n!}f^{(n)}(a)+\ldots \tag{6.15}$$

3. Putting $a+h=x$ or $h=x-a$, in (6.13), we have

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots \tag{6.16}$$

which expands $f(x)$ in ascending integral powers of $(x-a)$.

Example 6.4. Assuming the validity of expansion by Taylor's series, show that

$$\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\cdots\right).$$

Solution. Given, $f(x)=\sin x$. Let $a=\frac{\pi}{4}, h=\theta$, then by assuming the validity of expansion, we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots$$

or

$$\sin(a+h)=\sin a+h\cos a+\frac{h^2}{2!}(-\sin a)+\frac{h^3}{3!}(-\cos a)+\ldots$$

Putting $a=\frac{\pi}{4}$ and $h=\theta$, we obtain

$$\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}+\frac{\theta}{\sqrt{2}}+\frac{\theta^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)+\frac{\theta^3}{3!}\left(-\frac{1}{\sqrt{2}}\right)+\ldots$$

Hence,

$$\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\ldots\right).$$

Example 6.5. Assuming the validity of Taylor's series expansion, show that

$$\tan^{-1}x=\tan^{-1}\frac{\pi}{4}+\frac{\left(x-\frac{\pi}{4}\right)}{\left(1+\frac{\pi^2}{16}\right)}-\frac{\pi\left(x-\frac{\pi}{4}\right)^2}{4\left(1+\frac{\pi^2}{16}\right)^2}+\cdots$$

Solution. Given

$$f(x)=\tan^{-1}x$$

Then,

$$f'(x)=\frac{1}{1+x^2},$$

$$f''(x)=-\frac{2x}{(1+x^2)^2}\quad \text{etc.}$$

Now, $f(x)$ can be written as

$$f(x) = \tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4} + x - \frac{\pi}{4}\right) = \tan^{-1}(a+h),$$

where $a = \dfrac{\pi}{4}$, and $h = x - \dfrac{\pi}{4}$.

Assuming the validity of Taylor's series expansion, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots$$

$$\therefore \tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\frac{1}{1+\left(\frac{\pi^2}{16}\right)} - \frac{\pi}{4}\frac{\left(x-\frac{\pi}{4}\right)^2}{\left(1+\left(\frac{\pi^2}{16}\right)\right)^2} + \cdots$$

In-text Exercise 6.2. Solve the following questions:

1. Assuming the validity of Taylor's series expansion, show that

$$\sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \dots$$

2. Apply Taylor's series expansion to prove

$$\sec^{-1}(x+h) = \sec^{-1} x + \frac{h}{x\sqrt{x^2-1}} - \frac{(2x^2-1)}{x^2(x^2-1)^{3/2}}.\frac{h^2}{2!} + \dots$$

3. Apply Taylor's series expansion to prove

$$e^{1+h} = e\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots\right)$$

4. Expand $\tan x$ in power of $(x - \pi/4)$ up-to first four terms.

6.4.2 Maclaurin's Theorem

If in the statement of Taylor's Theorem 6.2, we put $a + h = x$ and $a = 0$, then we get the Maclaurin's Theorem as stated below:

Theorem 6.4. *If a function $f(x)$ is defined on $[0, x]$, such that*

(i) f and its derivatives $f', f'', f''', \dots, f^{(n-1)}$ are continuous on $[0, x]$.

(ii) $f^{(n)}(x)$ exist on $(0, x)$,

then there exists at least one point θ, $0 < \theta < 1$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} \cdot f^{(n-1)}(0) + \mathbf{R_n},$$

where $\mathbf{R_n} = \frac{x^n}{n!} f^{(n)}(\theta x)$, *(Lagrange's Remainder after n terms)*
and $\mathbf{R_n} = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x)$, *(Cauchy's Remainder after n terms).*

Example 6.6. Show that for every value of x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!} \sin \theta x, \qquad 0 < \theta < 1.$$

Solution. Let $f(x) = \sin x$. Then

$$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right) \quad n \in \mathbb{N}.$$

Now for $n = 2m$ (even)

$$f^{(n)}(x) = f^{(2n)}(x) = \sin\left(x + 2n \cdot \left(\frac{\pi}{2}\right)\right) = \sin(x + n\pi)$$
$$\implies f^n(0) = 0, \text{ when } n \text{ is even.}$$

Similarly, when $n = 2m - 1$ (odd), then

$$f^n(0) = (-1)^{\frac{n-1}{2}}$$
$$\text{or} \qquad f^{2n-1}(0) = (-1)^{n-1}$$
$$\text{Also} \qquad f^{2n}(\theta x) = \sin[\theta x + n\pi] = (-1)^n \sin(\theta x)$$

Therefore, by using Maclaurin's Theorem for $f(x) = \sin x$, we get

$$\therefore \qquad f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^{2n-1}}{(2n-1)!}f^{(2n-1)}(x) + \frac{x^{2n}}{(2n)!}f^{(2n)}(\theta x)$$

That is

$$\sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!}(-1)^1 + \ldots + \frac{x^{2n-1}}{(2n-1)!}(-1)^{n-1} + \frac{x^{2n}}{(2n)!}(-1)^n \sin \theta x$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n-1}}{(2n-1)!}(-1)^{n-1} + \frac{x^{2n}}{(2n)!}(-1)^n \sin \theta x.$$

6.4.3 Maclaurin's infinite series

Theorem 6.5 (Maclaurin's infinite series expansion). *If a function f defined on $[0, h]$ is such that*

(i) $f(x)$ possesses continuous derivatives of all order in $[0, h]$,

(ii) Taylor's remainder $\mathbf{R_n} \to 0$ *as* $n \to \infty$*,*

then for each $x \in [0, h]$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

This is called ***Maclaurin's infinite series expansion of*** $f(x)$***.***

6.5 Maclaurin's Series Expansion of Some Standard Functions

6.5.1 Maclaurin's Series Expansion of e^x

Let $f(x) = e^x \; x \in \mathbb{R}$, then

$$f'(x) = e^x, f''(x) = e^x, \ldots, f^{(n)}(x) = e^x \quad \forall n \in \mathbb{N} \tag{6.17}$$

$\implies f^{(n)}(x)$ exist for all $n \in \mathbb{N}$ and they are continuous as e^x is continuous.
Also,

$$f'(0) = e^0 = 1, f''(0) = e^0 = 1, \ldots, f^{(n)}(0) = e^0 = 1 \quad \forall n \in \mathbb{N} \tag{6.18}$$

The Lagrange's form of remainder R_n is given by

$$\begin{aligned} R_n &= \frac{x^n}{n!}f^{(n)}(\theta x), \quad 0 < \theta < 1 \qquad (6.19) \\ &= \frac{x^n}{n!}e^{\theta x} \quad 0 < \theta < 1. \\ &\to 0 \text{ as } n \to \infty \quad \text{as } \frac{x^n}{n!} = 0 \;\; \text{(from example 7.18)}. \end{aligned}$$

Thus, all the conditions of Maclaurin's Series Expansion are satisfied. Therefore, we have the expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Hence, substituting the value of the function and its derivatives from 6.18, we get

$$\begin{aligned} & e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \forall x \in \mathbf{R} \\ \text{or} \quad & e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

6.5.2 Maclaurin's Series Expansion of $\sin x$

Let $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \cos x = \sin\left(x + \frac{\pi}{2}\right), \\
f''(x) &= -\sin x = \sin(x + \pi), \\
f'''(x) &= -\cos x = \sin\left(x + \frac{3\pi}{2}\right), \\
&\vdots \\
f^{(n)}(x) &= \sin\left(x + \frac{n\pi}{2}\right), \quad \forall n \in \mathbb{N}. \qquad (6.20)
\end{aligned}$$

Therefore, $f(x)$ and its derivatives $f^{(n)}(x), n \in \mathbb{N}$ exist and they are continuous as $\sin x$ is continuous for all $x \in \mathbb{R}$. Also,

$$f(0) = \sin 0 = 0, f'(0) = \cos 0 = 1, f''(0) = -\sin 0 = 0, f'''(0) = -1 \cdots \qquad (6.21)$$

The Lagrange's form of remainder after n terms for $f(x) = \sin x$ is

$$\begin{aligned}
R_n &= \frac{x^n}{n!} f^{(n)}(\theta x), \quad 0 < \theta < 1 \qquad (6.22) \\
\implies R_n &= \left(\frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)\right) \qquad 0 < \theta < 1 \quad \text{(by using (6.20))} \\
\implies |R_n - 0| &= \left|\frac{x^n}{n!}\right| \cdot \left|\sin\left(\theta x + \frac{n\pi}{2}\right)\right| \\
&\leq \left|\frac{x^n}{n!}\right| \\
&\to 0 \text{ as } n \to \infty \quad \text{(by using example 7.18)}, \\
&\implies \lim_{n\to\infty} R_n = 0.
\end{aligned}$$

Thus, all the conditions of Maclaurin's Series Expansion are satisfied. Therefore, we have the expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Hence, substituting the value of the function and its derivatives, we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \forall x \in \mathbf{R}$$

$$\text{or} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

6.5.3 Maclaurin's Series Expansion of $\cos x$

Let $f(x) = \cos x$, then

$$f'(x) = -\sin x = \cos\left(x + \frac{\pi}{2}\right),$$
$$f''(x) = -\cos x = \cos(x + \pi),$$
$$f'''(x) = \sin x = \cos\left(x + \frac{3\pi}{2}\right),$$
$$\vdots$$
$$f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right), \quad \forall n \in \mathbb{N} \tag{6.23}$$

Therefore, $f(x)$ and its derivatives $f^{(n)}(x), n \in \mathbb{N}$ exist and they are continuous as $\cos x$ is continuous for all $x \in \mathbb{R}$. Also,

$$f(0) = \cos 0 = 1, f'(0) = -\sin 0 = 0, f''(0) = -\cos 0 = 1, f'''(0) = 0 \cdots \tag{6.24}$$

The Lagrange's form of remainder after n terms for $f(x) = \cos x$ is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \quad 0 < \theta < 1. \tag{6.25}$$

$$\implies R_n = \frac{x^n}{n!} \cdot \cos\left(\theta x + \frac{n\pi}{2}\right) \quad 0 < \theta < 1 \quad \text{(by using (6.23))}$$
$$|R_n - 0| = \left|\frac{x^n}{n!}\right| \cdot \left|\cos\theta x + \frac{n\pi}{2}\right|$$
$$\leq \left|\frac{x^n}{n!}\right|$$
$$\to 0 \text{ as } n \to \infty. \text{ (from exmaple 7.18)}.$$
$$\implies \lim_{n\to\infty} R_n = 0. \tag{6.26}$$

Thus, all the conditions of Maclaurin's Series Expansion are satisfied. Therefore, we have the expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Hence, substituting the values of the function and its derivatives, we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \forall x \in \mathbf{R}$$
$$\text{or} \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

6.5.4 Maclaurin's Series Expansion of $\log_e(1+x) \equiv ln(1+x)$

Let $f(x) = \log_e(1+x)$, then

$$f'(x) = \frac{1}{(1+x)}, f''(x) = \frac{-1}{(1+x)^2}, \ldots, f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \quad \forall n \in \mathbb{N} \text{ and } x > -1.$$

Therefore, $f(x)$ and its derivatives $f^{(n)}(x)$, $n \in \mathbb{N}$ are continuous for $x > -1$. Also,

$$f(0) = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 2, \ldots, f^{(n)}(0) = (-1)^{n-1}(n-1)! \quad \forall n \in \mathbb{N} \tag{6.27}$$

Now, we have two cases :

Case 1 : $0 \leq x \leq 1$

The Lagrange form of remainder after n terms is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) \tag{6.28}$$

$$\implies R_n = \frac{x^n}{n!} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} \quad \text{as} \quad f^{(n)}(\theta x) = \frac{(-1)^n(n-1)!}{(1+\theta x)^n}$$

$$= \frac{(-1)^{n-1}}{n} \left(\frac{x}{1+\theta x} \right)^n.$$

Since, $0 < \theta < 1$ and $0 \leq x \leq 1$, therefore $0 \leq \left(\frac{x}{(1+\theta x)} \right) < 1$. Therefore,

$$\left(\frac{x}{1+\theta x} \right)^n \to 0 \text{ as } n \to \infty \qquad (\text{using } \lim_{n\to\infty} (r^n) = 0, \text{ for } 0 < r < 1.)$$

Also,

$$\frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Hence,

$$R_n = \frac{(-1)^{n-1}}{n} \left(\frac{x}{1+\theta x} \right)^n \to 0 \text{ as } n \to \infty.$$

Case 2 : $-1 < x < 0$

For this case, we will consider Cauchy's form of remainder. That is

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x), \quad 0 < \theta < 1$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$= (-1)^{n-1} \left(\frac{1-\theta}{1+\theta x} \right)^{n-1} \cdot x^n \cdot \frac{1}{1+\theta x}.$$

Since $-1 < x < 0$ and $0 < \theta < 1$, therefore

$$-\theta < \theta x \implies 1 - \theta < 1 + \theta x, \text{ Also } 1 - \theta > 0$$

$$\implies 0 < \left(\frac{1-\theta}{1+\theta x}\right) < 1 \tag{6.29}$$

Therefore,

$$\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0 \text{ as } n \to \infty \text{ (using } \lim_{n\to\infty} r^n \to 0, \text{ as } n \to \infty \text{ for } 0 < r < 1.)$$

Also, $\quad -1 < x < 0 \quad \therefore x^n \to 0$ as $n \to \infty$. Therefore,

$$x^n \left(\frac{1-\theta}{1+\theta x}\right)^n \to 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$R_n = (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{1+\theta x} \to 0 \text{ as } n \to \infty.$$

Finally, combining **Case 1** and **Case 2**, we get,

$$R_n \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, both the condition of Maclaurin's Series Expansion are satisfied. Therefore, we have the expansion as

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots \\ \implies \log_e(1+x) &= 0 + x + \frac{x^2}{2!}(-1) + \frac{x^3}{3!} \cdot 1 + \dots + \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1} x^n}{n} + \dots \end{aligned}$$

or,

$$log_e(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

6.5.5 Maclaurin's Series Expansion of $(1+x)^m, \quad |x| < 1$

Let $f(x) = (1+x)^m$. We have the following two cases:

Case 1 : 'm' is a positive integer

Since

$$\begin{aligned} f(x) &= (1+x)^m, \\ f'(x) &= m(1+x)^{m-1}, \\ f''(x) &= m((m-1)(1+x)^{m-2}, \\ &\vdots \\ f^{(m)}(x) &= m!, \end{aligned}$$

$$\implies \qquad f^{(n)}(x) = 0 \qquad \forall \;\; n > m. \tag{6.30}$$

The Lagrange's form of remainder after n terms is

$$R_n = \frac{x^n}{n!} \cdot f^{(n)}(\theta x), \qquad 0 < \theta < 1.$$

For $n \to \infty$ we have $n > m$. Therefore $f^{(n)}(\theta x) = 0$ (using (6.30)). Also $\frac{x^n}{n!} \to 0$ as $n \to \infty$. (using 7.18). Thus, in this case we get

$$R_n \to 0 \text{ as } n \to \infty.$$

Hence, both the condition of Maclaurin's Series Expansion are satisfied in this case. Therefore

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^m}{m!} f^m(0) \\ &= 1 + xm + \frac{x^2}{2!} m(m-1) + \ldots + \frac{x^m}{m!} m! \\ &= 1 + mx + \frac{m(m-1)x^2}{2!} + \ldots + x^m. \end{aligned}$$

Thus, if m, is a fixed positive integer, then we get a finite series expansion of $(1+x)^m$.

Case 2 : 'm' is not a positive integer.

We have,

$$\begin{aligned} f(x) &= (1+x)^m \\ f'(x) &= m(1+x)^{m-1} \\ f''(x) &= m((m-1)(1+x)^{m-2} \\ &\vdots \\ f^{(n)}(x) &= m(m-1)(m-2)\ldots(m-n+1)(1+x)^{m-n} \qquad \forall n \in \mathbb{N} \end{aligned}$$

Also,

$$f(0) = 1, \quad f'(0) = m, \quad f''(0) = m(m-1), \quad \ldots, f^{(n)}(0) = m(m-1)\ldots(m-n+1)$$

Here, we will use Cauchy's form of remainder. Therefore

$$\begin{aligned} R_n &= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} \cdot f^n(\theta x) \\ &= x^n \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-m}} \cdot \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} \\ &= x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1} \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} \end{aligned} \tag{6.31}$$

Since,

$$|x| < 1, \qquad \therefore x^n \to 0 \text{ as } n \to \infty. \tag{6.32}$$

Also, from equation (6.29)

$$0 < \frac{1-\theta}{1+\theta x} < 1$$

Therefore

$$\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0 \text{ as } n \to \infty \tag{6.33}$$

Also,

$$\text{for } m > 1, (1+\theta x)^{m-1} < (1+|x|)^{m-1} \text{ and}$$
$$\text{for } m < 1, (1+\theta x)^{m-1} < (1-|x|)^{m-1} \tag{6.34}$$

$\implies (1+\theta x)^{m-1}$ is a finite real number. Thus, using above equations 6.32, 6.33, 6.34 in 6.31, we get

$$R_n \to 0 \text{ as } n \to \infty \quad \text{for} \quad |x| < 1$$

Hence, both the condition of Maclaurin's Series Expansion are satisfied. Therefore, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$
$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots \frac{m(m-1)(m-n+1)}{n!}x^n + \ldots$$

Thus, for non positive integral value of 'm', we get a infinite series expansion of $(1+x)^m$.

Example 6.7. Assuming the validity of expansion by Maclaurin's series, prove that

$$e^n \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \cdots$$

Solution. Since Maclaurin's expansion is valid for the function $f(x) = e^x \cos x$, therefore

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots \tag{6.35}$$

We have

$$\begin{aligned} f(x) &= e^x \cos x, \\ f'(x) &= e^x \cos x - e^x \sin x \\ &= \sqrt{2}\, e^x \left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) \\ &= \sqrt{2}\, e^x \cos\left(x + \frac{\pi}{4}\right) \end{aligned}$$

Similarly, we get

$$f^n(x) = 2^{\frac{n}{2}} e^x \cos\left(x + \frac{n\pi}{4}\right), \; n \in \mathbb{N}.$$

Also,

$$f(0) = 1, f'(0) = 1, f''(0) = 0, f'''(0) = -2, f^{iv}(0) = -4, f^{v}(0) = -4 \quad \text{etc.}$$

Substituting these values of $f(0), f'(0), f''(0), f'''(0), f^{iv}(0)$ and $f^{v}(0)$ in equation (6.35), we get

$$e^{x}\ \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \cdots$$

Example 6.8. Assuming the validity of expansion, expand a^x and e^x in powers of x by Maclaurin's theorem.

Solution. Let

$$\begin{aligned}
f(x) &= a^x && \Longrightarrow f(0) = 1\\
f'(x) &= a^x \log a && \Longrightarrow f'(0) = \log a\\
f''(x) &= a^x (\log a)^2 && \Longrightarrow f''(0) = (\log a)^2\\
f'''(x) &= a^x (\log a)^3 && \Longrightarrow f'''(0) = (\log a)^3
\end{aligned}$$

and so on. Therefore, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$\text{Therefore,} \quad a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots \tag{6.36}$$

Putting $a = e$ and $\log a = \log e = 1$ in equation (6.36), we get the Maclaurin's series expansion of e^x as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

In-text Exercise 6.3. Solve the following questions:

1. Assuming the validity of expansion by Maclaurin's series, show that

   (i) $\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \ldots$

   (ii) $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$

2. Expand by Maclaurin's theorem $\dfrac{e^x}{e^x + 1}$ as far as the term containing x^3.

3. Show that by means of Maclaurin's series expansion, that

$$\log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$$

4. Assuming the validity of Maclaurin's series expansion, find the series expansion of $f(x) = e^{2x}$ for all real values of x.

6.6 Summary

In this lesson, we have discussed following topics:

1. Cauchy's Mean Value Theorem: Let there be two functions,$f(x)$ and $g(x)$. These two functions shall be continuous on the interval, $[a, b]$, and these functions are differentiable on the range (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there will be a point $x = c$ in the given range or the interval such that, $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$

2. Taylor's Theorem with Lagrange's Form of Remainder: If a function f is defined on $[a, a + h]$, such that

   (i) f and the derivatives f', f'',..., $f^{(n-1)}$ are continuous on $[a, a + h]$,

   (ii) the n^{th} derivative exist on $(a, a + h)$,

   then there exists at least one point θ, $0 < \theta < 1$ such that

   $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h).$$

   The $(n + 1)th$ term i.e. $\dfrac{h^n}{n!}f^{(n)}(a + \theta h)$ is called the Lagrange's remainder after n terms and is denoted by $\mathbf{R_n}$.

3. Taylor's Theorem with Cauchy's Form of Remainder: If the remainder $\mathbf{R_n}$ is expressed as

   $$\mathbf{R_n} = \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h), \tag{6.37}$$

   then the above theorem is called **Taylor's Theorem with Cauchy's Form of Remainder** and $\mathbf{R_n}$ is called Cauchy's Remainder after n terms.

4. Taylor's series Expansions of Functions: If a function $f(x)$ is such that

   (i) $f(x)$ possesses continuous derivatives of all orders in $[a, a + h]$.

   (ii) For $0 < \theta < 1$, Taylor's remainder $R_n = \frac{h^n}{n!}f^{(n)}(a + \theta h)$ *tends to* 0 *as* $n \to \infty$,

   then

   $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \ldots. \tag{6.38}$$

5. Maclaurin's series Expansions of Functions: If a function f defined on $[0, h]$ is such that

   (i) $f(x)$ possesses continuous derivatives of all order in $[0, h]$,

   (ii) Taylor's remainder $\mathbf{R_n} \to 0$ as $n \to \infty$, then for each $x \in [0, h]$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

This is called **Maclaurin's infinite series expansion of** $f(x)$**.**

6.7 Self-Assessment Exercises

1. Verify Cauchy's Mean Value Theorem for the following functions:

$$f(x) = x(x-1)(x-2), g(x) = x(x-2)(x-3), a = 0, b = \frac{1}{2}.$$

2. Check the validity of Cauchy's Mean Value Theorem for the following functions:

$$f(x) = x^4, g(x) = x^2, a = 1, b = 2.$$

3. Show that

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$$

4. For $-1 < x \leq 1$, show that

$$\log(1-x) = -x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - ...$$

5. If $f(x) = x^3 + 2x^2 - 5x + 11$, find the value of $f(\frac{9}{10})$ with the help of Taylor's series for $f(x+h)$.

6. Apply Maclaurin's theorem to prove

$$\cos^2 x = 1 - x^2 + \frac{x^4}{3} + ...$$

7. prove $e^{ax}\cos bx$ is equal to

$$1+ax+(a^2-b^2)\frac{x^2}{2!}+a(a^2-3b^2)\frac{x^3}{3!}+...+\frac{x^n}{n!}(a^2+b^2)^{n/2}e^{a\theta x}\cos\left(b\theta x + n\tan^{-1}\frac{b}{a}\right).$$

8. Expand $e^{ax}\sin bx$ by Maclaurin's theorem with Cauchy's form of remainder.

6.8 Solutions to In-text Exercises

Exercise 6.1

1. (i) $c = -\frac{\pi}{4} \in (-\pi/2, 0)$.
   (ii) $c = \frac{1}{2} \in (0, 1)$.

2. Let $g(x) = x^2$, then using Cauchy's Mean Value Theorem given result can be obtained.

Exercise 6.2

1. $f(x) = \sin x = \sin\left(\frac{\pi}{2} + x - \frac{\pi}{2}\right) = \sin(a + h)$, where $a = \pi/2$ and $h = x - \pi/2$.
2. Using Taylor expansion, we can obtained the required expansion.
3. $e^{1+h} = e(e^h)$
4. $\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + ...$

Exercise 6.3

1. (i) By taking $f(x) = \log \sec x$ and substituting in Maclaurin's expansion, required result can be obtained.
   (ii) Use the expansion of $\log(1 + x)$ to prove it.
2. Expansion is $\frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + ...$
3. Use the expansion of $\log(1 + x)$ to prove it.
4. Expansion is $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + ...$

6.9 Suggested Readings

1. Narayan, Shanti (Revised by Mittal, P. K.). Differential Calculus. S. Chand, Delhi, 2019.
2. Prasad, Gorakh (2016). Differential Calculus (19th ed.) Pothishala Pvt. Ltd. Allahabad.
3. Thomas Jr., George B., Weir, Maurice D.,Hass, Joel (2014). Thomas Calculus (13th ed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 7

Extremum and Convergence of Series

Structure

7.1 Learning Objectives

The learning objectives of this lesson are to:

- use differentiation to locate the stationary points of a function.

- distinguish whether these stationary points are the points of maxima, minima or the points of inflexion.
- understand the difference between local and global maxima and minima.
- to impart the knowledge of convergence of sequences and summation of series.
- to understand the concept of the sequence of partial sums in order to understand the convergence of series.

7.2 Introduction

In lesson 5, we have learnt about various applications of differentiation. In this chapter, we will use differentiation to find the maximum and minimum values of differentiable functions useful for solving some applied problems. The terms maxima and minima refer to extreme values of a function, that is, the maximum and minimum values that the function attains. Also, we will learn here about convergence of sequence and series of real numbers.

7.3 Extremum of a Function

Let $y = f(x)$ be a function of real variable defined in an interval $[a, b]$. The extremum of $f(x)$ in $[a, b]$ is the extreme value of $f(x)$ in $[a, b]$. That is, it is either the maximum value (maxima) or the minimum value (minima) of $f(x)$ in $[a, b]$. Geometrically, maxima and minima of a function are its peaks and valleys, as shown in the following figure 7.3.

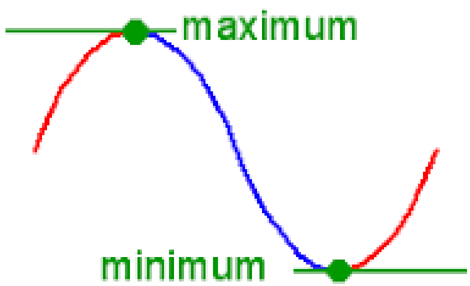


Figure 7.1: Maxima and Minima of a function.

The maxima and minima of a function are of two types,

1. Local Maxima and Local Minima
2. Absolute (Global) Maxima and Absolute (Global) Minima

7.3.1 Local Maxima and Local Minima

Local maxima and minima are the maxima and minima of the function which arise in a **particular interval**. Local maxima would be the value of a function at a point in a particular interval for which the values of the function near that point are always less than the value of the function at that point. Whereas local minima would be the value of the function at a point where the values of the function near that point are greater than the value of the function at that point. It is possible for a function to have as many local maxima and minima as it needs.

Definition 7.1. (Local Maxima) Let $y = f(x)$ be a function defined on $[a, b]$. Then $f(x)$ is said to attain a local maximum at $x = c$, if there exist a neighborhood of c $(c-\delta, c+\delta) \subseteq [a, b]$ such that

$$f(x) \leq f(c) \ \ \forall \ \ x \in (c-\delta, c+\delta).$$

In this case, $f(c)$ is called the local maximum value of $f(x)$ at $x = c$.

Definition 7.2. (Local Minima) Let $y = f(x)$ be a function defined on $[a, b]$. Then $f(x)$ is said to attain a local minimum at $x = c$ if there exist a neighborhood of c $(c-\delta, c+\delta) \subseteq [a, b]$, such that

$$f(x) \geq f(c) \ \ \forall \ \ x \in (c-\delta, c+\delta).$$

In this case, $f(c)$ is called the local minimum value of $f(x)$ at $x = c$.

Note.
1. A local maxima (or local minima) is also known as a relative maxima (or relative minima).
2. A function $f(x)$ defined in a given domain $[a, b]$ may possess many local maxima and local minima as shown in the following figure 7.3.1.

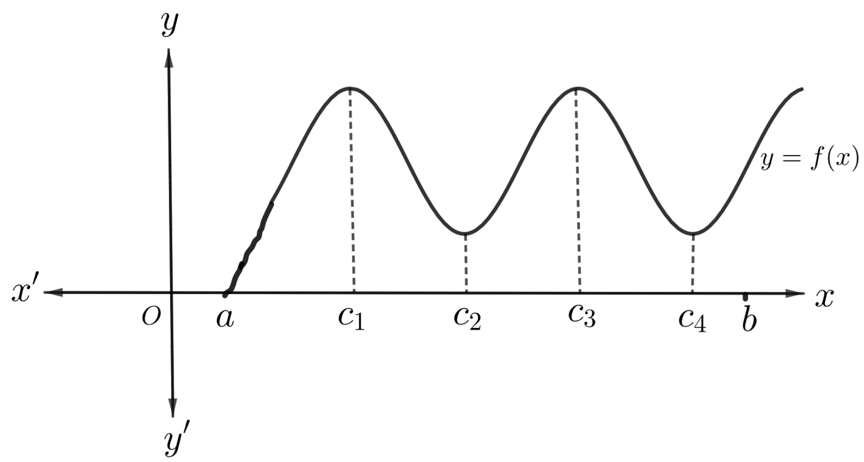


Figure 7.2: Local maxima at c_1, c_3 and Local minima at c_2 and c_4.

7.3.2 Global Maxima and Global Minima

The highest point of a function within the **entire domain** is known as the absolute maxima of the function whereas the lowest point of the function within the entire domain of the function, is known as the absolute minima of the function. There can only be one absolute maximum of a function and one absolute minimum of the function over the entire domain. The absolute maxima and minima of the function are also called as the global maxima and global minima of the function. The Global (Absolute) maxima of a function $f(x)$ defined on $[a, b]$ is the greatest value of $f(x)$ in $[a, b]$. Similarly, the Global (Absolute) minima of a function $f(x)$ defined on $[a, b]$ is the least value of $f(x)$ in $[a, b]$. Precisely, we have the following definitions:

Definition 7.3. (Absolute Maxima) Let $f(x)$ be a real function defined on an interval $I = [a, b]$. Then, $f(x)$ is said to attain the global maximum at $x = c$, if

$$f(x) \leq f(c) \;\; \forall \;\; x \in I.$$

Here, $f(c)$ is called the global maximum value of $f(x)$ in the interval I.

Definition 7.4. (Absolute Minima) Let $f(x)$ be a real function defined on an interval $I = [a, b]$. Then, $f(x)$ is said to attain the global minimum at $x = c$, if

$$f(x) \geq f(c) \;\; \forall \;\; x \in I.$$

Here, $f(c)$ is called the global minimum value of $f(x)$ in the interval I.

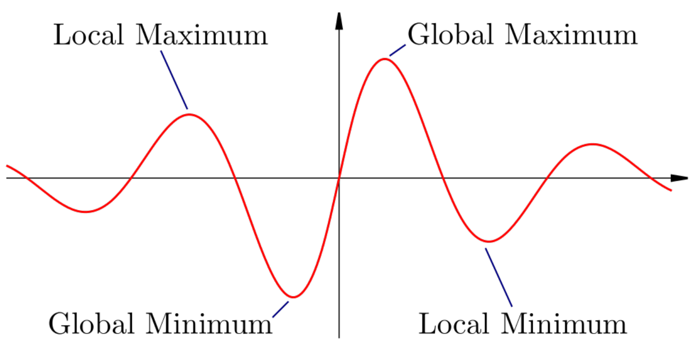


Figure 7.3: (Global maxima and Global minima of a function.)

Note. The Global extrema of a function $f(x)$ defined on $[a, b]$ occurs either at the points of local extrema or at the end points a, b of $[a, b]$.

7.3.3 A Necessary condition for Local Extrema

Let $y = f(x)$ be a differentiable function on (a, b). If $f(x)$ has a local extremum at $x = c \in (a, b)$, then $f'(c) = 0$.

Note. (i) The necessary condition stated above holds for a differentiable function. However, a local extremum may occur at a point c at which $f(x)$ is not differentiable (see Figure 7.3.4).

(ii) A point $c \in [a, b]$, such that $f'(c) = 0$ or $f'(c)$ does not exist, is called a **critical point**. If $f'(c) = 0$, then c is called a **stationary point**.

(iii) The condition cited above is not a sufficient condition.

Example 7.1. Let $f(x) = x^3$. Then at $x = 0$ function is differentiable and derivative is 0 but its is neither a point of maxima nor a point of as shown in figure 7.3.3.

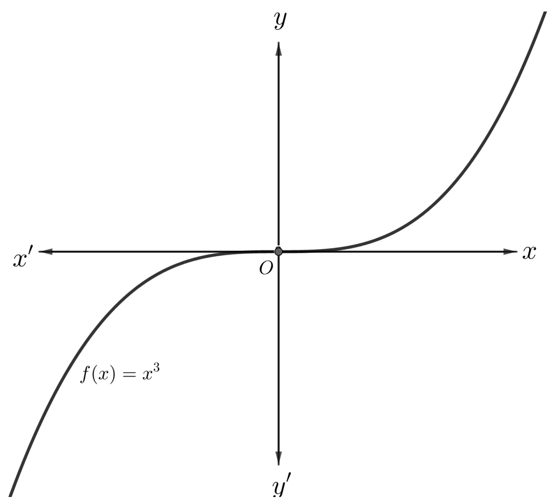


Figure 7.4: (Graph of $y = x^3$)

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7.3.4 How to find Maxima and Minima of a Function

We have the following tests to find the local extrema of a differentiable function.

1. First Order Derivative Test
2. Second-Order Derivative Test

First Order Derivative Test

The first order derivative test gives a sufficient condition for $f(x)$ to have local extremum at $x = c$. It is stated as following:

Let $f(x)$ be a differentiable function on (a, b). Then

(i) if $f'(x)$ changes sign from positive to negative as x passes through c from left to right, then $f(x)$ has a local maximum at $x = c$.

(ii) if $f'(x)$ changes sign from negative to positive as x passes through c from left to right, then $f(x)$ has a local minimum at $x = c$.

(iii) if $f'(x)$ does not changes sign as x passes through c, then $f(x)$ has no local extrema at $x = c$.

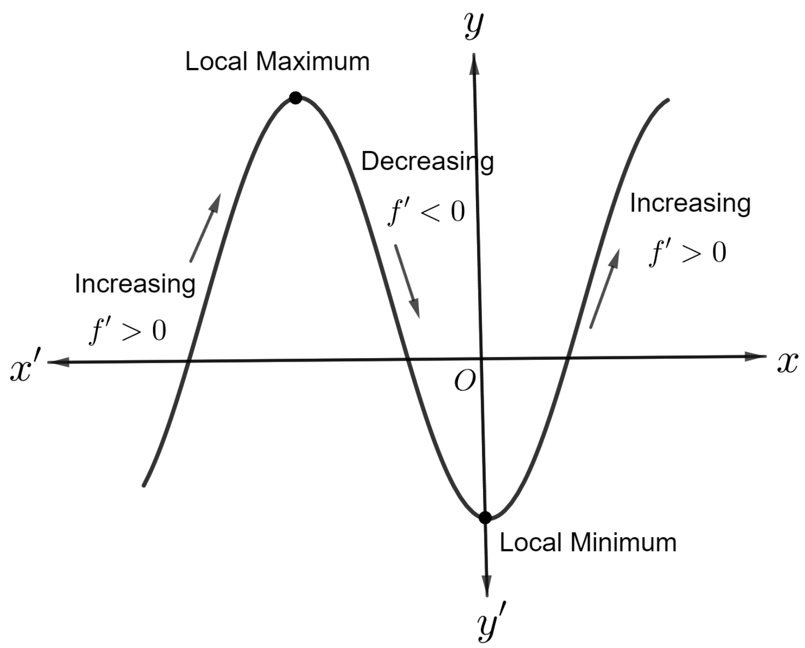


Figure 7.5: Graph for first derivative test.

Algorithm for finding the Local Maxima and Local Minima of a Function using first derivative test

Step-1 : For the function $y = f(x)$, find $\dfrac{dy}{dx} = f'(x)$.

Step-2 : Put $\dfrac{dy}{dx} = 0$ and solve this equation for x. Let $c_1, c_2, ..., c_n$ be the roots of this equation, then these points are the stationary points.

Step-3 : Choose one stationary point c_1 and check the change in the sign of the function as x passes through c from left to right.

(a) If $\dfrac{dy}{dx}$ changes its sign from positive to negative as x passes through c_1, then the function attains a local maximum at $x = c_1$.

(b) If $\dfrac{dy}{dx}$ changes its sign from negative to positive as x passes through c_1, then the function attains a local minimum at $x = c_1$.

Step-4 : Repeat the process for all other vales of $x = c_2, c_3, ..., c_n$.

Example 7.2. Find all the points of local maxima and minima of the function

$$f(x) = x^3 - 6x^2 + 9x - 8.$$

Hence, find the corresponding local maximum and minimum values.

Solution. Let $y = f(x) = x^3 - 6x^2 + 9x - 8$. Then,

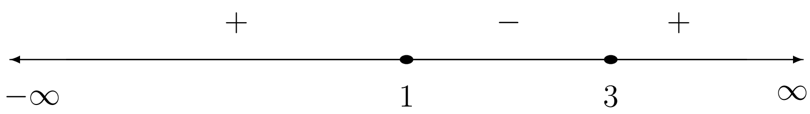
$$\frac{dy}{dx} = f'(x) = 3(x^2 - 4x + 4) = 3(x-1)(x-3).$$

The stationary points of $f(x)$ are given by $f'(x) = 0$. Thus,

$$\frac{dy}{dx} = f'(x) = 0 \Rightarrow x = 1, 3.$$

Now,

$$\begin{aligned}
&\text{if } x < 1, \text{ then } (x-1) < 0 \text{ and } x - 3 < 0 \implies f'(x) > 0;\\
&\text{if } 1 < x < 3, \text{ then } (x-1) > 0 \text{ and } x - 3 < 0 \implies f'(x) < 0 \text{ and}\\
&\text{if } x > 3, \text{ then } (x-1) > 0 \text{ and } x - 3 > 0 \implies f'(x) > 0.
\end{aligned}$$



Signs of $f'(x)$ for different values of x

Clearly, $f'(x)$ changes sign from positive to negative as x passes through 1. So, $x = 1$ is point of local maxima. The corresponding local maximum value is $f(1) = -4$.

Also, $f'(x)$ changes sign from negative to positive as x passes through 3. So, $x = 3$ is a point of local minimum and the corresponding local minimum value is $f(3) = -8$.

Example 7.3. Find the points at which f given by

$$f(x) = (x-2)^4(x+1)^3$$

has (i) local maxima (ii) local minima (iii) no local extremum.

Solution. We have,

$$\begin{aligned}
&f(x) = (x-2)^4(x+1)^3\\
&\Rightarrow f'(x) = 4(x-2)^3(x+1)^3 + 3(x-2)^4(x+1)^2\\
&\Rightarrow f'(x) = (x-2)^3(x+1)^2(7x-2).\\
&\Rightarrow f'(x) = (x-2)^2(x+1)^2(x-2)(7x-2).
\end{aligned}$$

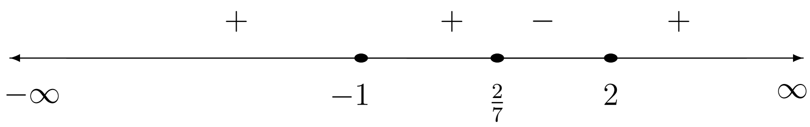
Now, for stationary points

Now,

$$f'(x) = 0 \Rightarrow x = -1,\ 2,\ \frac{2}{7}.$$

$\implies$ The stationary points are $x = -1, 2$ and $\frac{2}{7}$. Since $(x-2)^2(x+1)^2$ is always positive, therefore the sign of $f'(x)$ depends upon the sign of $(x-2)(7x-2)$. The changes in sign of $f'(x)$ as x increases through -1, $\frac{2}{7}$ and 2 are given by

$$\begin{aligned} f'(x) &> 0 \ \text{ if } \ x < -1, \\ f'(x) &> 0 \ \text{ if } \ -1 < x < \frac{2}{7}, \\ f'(x) &< 0 \ \text{ if } \ \frac{2}{7} < x < 2, \\ \text{and } f'(x) &> 0 \ \text{ if } \ x > 2. \end{aligned}$$



Signs of $f'(x)$ for different values of x.

Therefore, $f'(x)$ changes its sign from positive to negative as x passes through $\frac{2}{7}$. So, $x = \frac{2}{7}$ is a point of local maximum.

$f'(x)$ changes its sign from negative to positive as x passes through 2. So, $x = 2$ is a point of local minimum.

Since, there is no change in the sign of $f'(x)$ as increases through -1. Therefore, no local extremum exist at $x = -1$.

In-text Exercise 7.1. Solve the following questions:

1. Find all the points of local maxima and local minima of $f(x) = x^3 - 6x^2 + 12x - 8$.

2. Find the local maxima and local minima of the function $f(x) = \sin x + \cos x,\ \ 0 < x < \frac{\pi}{2}$ using the first derivative test.

3. Find all the points of local maxima and local minima of $f(x) = \cos x,\ \ 0 < x < \pi$ using the first derivative test.

4. Find the points at which the function $f(x) = (x-1)(x+2)^2$ has (i) local maxima (ii) local minima (iii) no local extremum.

Second-Order Derivative Test

Let $y = f(x)$ be a function defined on (a, b) and $f'(x)$ exists at each $x \in (a, b)$. Then and it is two times differentiable at at a point $c \in I$. Then

1. $f(x)$ has a local maximum at $x = c \in (a, b)$, if $f'(c) = 0$ and $f''(c) < 0$.
2. $f(x)$ has a local minimum at $x = c \in (a, b)$, if $f'(c) = 0$ and $f''(c) > 0$.
3. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

Example 7.4. Examine the following function for local maximum and minimum values

$$f(x) = x^5 - 5x^4 + 5x^3 - 1.$$

Solution. Given

$$\begin{aligned} & f(x) = x^5 - 5x^4 + 5x^3 - 1. \\ \implies & f'(x) = 5x^4 - 20x^3 + 15x^2. \end{aligned}$$

Now,

$$\begin{aligned} & f'(x) = 0 \\ \implies & 5x^4 - 20x^3 + 15x^2 = 0 \\ \implies & 5x^2(x-1)(x-3) = 0 \\ \implies & x = 0, 1, 3. \end{aligned}$$

Therefore, the stationary (critical) points are $x = 0$, 1 and 3. Now,

$$f''(x) = 20x^3 - 60x^2 + 30x$$

For $x = 1$, $\quad f''(1) = 20 - 60 + 30 = -10 < 0$.
For $x = 3$, $\quad f''(3) = 20(3)^3 - 60(3)^2 + 30(3) = 540 - 540 + 90 = 90 > 0$.

Therefore by the second order derivative Test, $x = 1$ is a point of local maximum and $x = 3$ is a point of local minimum. Also $f''(0) = 0$, therefore the second order derivative test fails for $x = 0$.

In-text Exercise 7.2. Solve the following questions:

1. Find all the points of local maxima and minima of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$. Also, find the corresponding maximum and minimum values.
2. Show that the function $f(x) = x^3 + x^2 + x + 1$ doesn't has a point of local maxima and local minima.
3. Find the points of local maxima and minima for the following functions

   (i) $f(x) = (x - 1)(x + 2)^2$.

(ii) $f(x) = x + \sqrt{1-x},\ x \leq 1.$

(iii) $f(x) = \sin x + \cos x, \quad 0 < x < \frac{\pi}{2}.$

(iv) $f(x) = 2\cos x + x, \quad 0 < x < \pi.$

4. Find the maximum profit that a company can make, if the profit function is given as $P(x) = 24x - 18x^2 + 41$.

5. If $f(x) = a \log |x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find the value of constants a and b.

6. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\left(\frac{1}{e}\right)}$.

7.3.5 Absolute Maximum and Absolute Minimum in a closed interval

Let $y = f(x)$ be a function defined on a closed interval $[a, b]$. Let $f'(x)$ exists and it is continuous at $x \in (a, b)$. In order to find the Absolute (Global) extrema of $f(x)$ in $[a, b]$, we first find the local extrema of $f(x)$ in (a, b) and then find $f(a)$ and $f(b)$. Then

(i) the absolute maximum of $f(x)$ in $[a, b]$ = max { local maxima of $f(x)$ in (a, b), $f(a)$, $f(b)$. }

(ii) the absolute minimum of $f(x)$ in $[a, b]$ = min { local minima of $f(x)$ in (a, b), $f(a)$, $f(b)$. }

Example 7.5. Find the largest and smallest values of the polynomial $x^3 - 18x^2 + 96x$ in the interval $[0, 9]$.

Solution. The given polynomial function

$$f(x) = x^3 - 18x^2 + 96x$$

is differentiable on $(0, 9)$ and

$$f'(x) = 3x^2 - 36x + 96.$$

For the critical points we have

$$\begin{aligned} f'(x) = 3x^2 - 36x + 96 &= 0 \\ \implies \quad (x-8)(x-4) &= 0 \\ \implies \quad x = 4, 8 &\in [0, 9]. \end{aligned}$$

Therefore, $x = 4, 8$ are the points of local maxima and local minima. Now, to find the absolute extrema of $f(x)$ in $[0, 9]$, we consider the values of $f(x)$ at the points of local extrema (i.e. $x = 4, 8$) as well as the values $f(a)$ and $f(b)$ at the end points $a = 0, b = 9$.

$$\begin{aligned} &f(0) = 0 \\ &f(4) = (4)^3 - 18(4)^2 + 96(4) = 160 \\ &f(8) = (8)^3 - 18(8)^2 + 96(8) = 128 \\ &f(9) = (9)^3 - 18(9)^2 + 96(9) = 135. \end{aligned}$$

Therefore, the absolute minima is smallest value of the given polynomial occurring at $x = 0$ and the largest value of the given polynomial is occurring at $x = 4$. Thus the largest value is 160 and smallest value is 0.

In-text Exercise 7.3. Solve the following questions:

1. Find the maximum (largest) and minimum (smallest) values of $f(x) = \sin x$ in the interval $[\pi, 2\pi]$.
2. Find the absolute maximum and absolute minimum values of the function $f(x) = x^2 - 2x + 4 = 0$ in the interval $[-3, 1]$.
3. Find both the maximum and minimum values of the $f(x) = 2x^3 - 15x^2 + 36x + 1$ on the interval $[1, 5]$.
4. Find the global extrema of the given function $f(x) = x + \sin 2x$ in the interval $[0, 2\pi]$.

7.3.6 Applications of Maxima and Minima

In the following section, we shall apply the theory of maxima and minima to solve practical problems involving the use of the same. For example to maximize the area, volume, profit etc.

Example 7.6. Show that all the rectangles with a given perimeter, the square has the largest area.

Solution. Let x and y be the lengths of two sides of the rectangle of fixed parameter P, and let A be its area. Then,

$$P = 2(x + y) \tag{7.1}$$

and

$$A = xy \tag{7.2}$$

Substituting the value of y from (7.1) into (7.2), we get

$$A = xy = x\left(\frac{P}{2} - x\right) = \left(\frac{Px}{2} - x^2\right)$$

$$\implies \frac{dA}{dx} = \left(\frac{P}{2} - 2x\right) \text{ and } \frac{d^2A}{dx^2} = -2.$$

The critical points of A are given by $\dfrac{dA}{dx} = 0$.

$$\therefore \left(\frac{P}{2} - 2x\right) = 0 \implies P = 4x \implies 2x + 2y = 4x \implies x = y.$$

Also,

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ at } x = y.$$

Hence A is maximum when $x = y$ i.e. the rectangle is a square.

Example 7.7. Show that the height of an cylinder of given surface and greatest volume is equal to the radius of its base.

Solution. Let r be the radius of the circular base, h be the height, S be the surface and V the volume of the cylinder. Therefore,

$$S = \pi r^2 + 2\pi r h, \tag{7.3}$$

and

$$V = \pi r^2 h \tag{7.4}$$

Since surface is given, there S is constant and V is a variable. Also, h, r, are variables. Substituting the value of h, as obtained from (7.3), in (7.4), we get

$$V = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) = \left(\frac{Sr - \pi r^3}{2}\right), \tag{7.5}$$

which gives V in terms of single variable r. Now,

$$\frac{dV}{dr} = \left(\frac{S - 3\pi r^2}{2}\right), \tag{7.6}$$

is 0 when $r = \sqrt{\frac{S}{3\pi}}$. Thus V has only one stationary value. As V must be positive, we have

$$Sr - \pi r^3 > 0 \text{ i.e. } Sr > \pi r^3 \text{ or } \; r < \sqrt{\frac{S}{\pi}}.$$

Thus r varies in the interval $(0, \sqrt{\frac{S}{\pi}})$. Now $V = 0$ for the points $r = 0$ and $\sqrt{\frac{S}{\pi}}$ and is positive for every other admissible value of x. Hence V is greatest for $r = \sqrt{\frac{S}{3\pi}}$.

Substituting this value of r in (7.3), we get

$$\begin{aligned} h = \frac{S - \pi r^2}{2\pi r} &= \frac{S - \pi \cdot \frac{S}{3\pi}}{2\pi\sqrt{\frac{S}{3\pi}}} \\ &= \sqrt{\frac{S}{3\pi}} \end{aligned}$$

Hence, for a cylinder of greatest volume and given surface $h = r$.

7.4 Sequences

Sequences occur frequently in analysis, and they appear in many contexts. While we are all familiar with sequences, it is useful to have a formal definition.

Definition 7.5. A sequence of real number is defined as a function $F : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of natural number and $\mathbb{R}$ is the set of real numbers. A sequence may be written as

$$< f_1, f_2, f_3..., f_n, ... > \quad \text{or} \quad < f_n > \quad \text{or} \quad (f_n).$$

The real numbers $f_1, f_2, f_3..., f_n, ...$ are called the terms or elements of the sequence. f_1 is called the first term, f_2 is called the second term,...,f_n is called the nth term of the sequence $< f_n >$. Analogous definitions can be given for the sequence of natural numbers, integers, etc. In this lesson, we shall consider only sequences of real numbers.

Example 7.8. Following are the sequences of real numbers:

1. $< n >=< 1, 2, 3, 4, ... >$
2. $< n^2 >=< 1, 4, 9, 16, ... >$
3. $< (-1)^n >=< -1, 1, -1, 1, ... >$
4. $< \dfrac{n}{n+1} >=< \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, ... >$
5. $< 1 + (-1)^n >=< 0, 2, 0, 2, ... >$

From the above examples, we can observe that in a sequence all the terms can be distinct or repeating. Also the sequence has always an infinite number of elements.

Definition 7.6 (Range of a sequence)**.** The set of all distinct elements of a sequence is called the range set of the given sequence. The range set of the sequence $< a_n >$ is the set $\{a_n : n \in \mathbb{N}\}$.

Example 7.9. Range sets of the sequences in Example 7.8 are:

1. $\{1, 2, 3, 4, ...\}$
2. $\{1, 4, 9, 16, ...\}$
3. $\{-1, 1\}$
4. $\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, ...\}$
5. $\{0, 2\}$

Thus, it can be observed that the range set of a sequence may be finite or infinite.

7.4.1 Bounded sequence

Definition 7.7. A sequence $< a_n >$ is said to be **bounded above**, if there exists a real number M such that $a_n \leq M \ \forall \ n \in \mathbb{N}$ and the real number M is called an **upper bound** of the sequence $< a_n >$.

Definition 7.8. A sequence $< a_n >$ is said to be **bounded below**, if there exists a real number M_0 such that $a_n \geq M_0 \ \forall \ n \in \mathbb{N}$ and the real number M_0 is called an **lower bound** of the sequence $< a_n >$.

Definition 7.9. A sequence which is bounded above as well as bounded below is called a bounded sequence. Eventually, $< a_n >$ is bounded if there exist two real numbers M_0 and M such that

$$M_0 \leq a_n \leq M \ \forall \ n \in \mathbb{N}$$

Example 7.10. 1. The sequence $< n >=< 1, 2, 3, 4, ... >$ is bounded below by 1 but it is not bounded above.

2. The sequence $< n^2 >=< 1, 4, 9, 16, ... >$ is bounded below by 1 but not bounded above.

3. The sequence $< (-1)^n >=< -1, 1, -1, 1, ... >$ is bounded, -1 being a lower bound and 1 is an upper bound.

4. The sequence $< \dfrac{n}{n+1} >=< \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, ... >$ is bounded below by $\dfrac{1}{2}$ but it is not bounded above.

5. The sequence $< 1 + (-1)^n >=< 0, 2, 0, 2... >$ is bounded below by 0 and bounded above by 2.

7.4.2 Convergence of a sequence

A fundamental concept in mathematics is that of convergence. Consider the sequences listed in Example 7.8 and observe the way how a sequence $< a_n >$ vary as n becomes larger and larger.

Example 7.11. 1. $< n >=< 1, 2, 3, 4, ... >$. In this sequence, the terms becomes larger and larger and tends to $+\infty$ as $n \to +\infty$.

2. $< n^2 >=< 1, 4, 9, 16, ... >$. In this sequence also the terms becomes larger and larger and tends to $+\infty$ as $n \to +\infty$.

3. $< \dfrac{n}{n+1} >=< \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, ... >$. In this sequence the terms come closer and closer to 1 as n becomes larger and larger. We write $< \dfrac{n}{n+1} > \to 1$ as $n \to \infty$.

4. $< 1 + (-1)^n > = < 0, 2, 0, 2... >$. In this sequence, the terms of the sequence oscillates with values 0 and 2, and does not come closer to any number as n becomes larger and larger.

Now, we make the precise definition of a convergent sequence of real numbers.

Definition 7.10. A sequence $< a_n >$ in $\mathbb{R}$ is said to converge to a real number a if for every $\epsilon > 0$, there exists positive integer k (in general depending on ϵ) such that

$$|a_n - a| < \epsilon, \;\; \forall \;\; n \geq k.$$

The number a is then called the limit of the sequence $< a_n >$ and $< a_n >$ is called a convergent sequence.

Note. 1. If $< a_n >$ converges to a, then we denote the convergence by writing $\lim\limits_{n\to\infty} < a_n >= a$, or $< a_n >\to a$ as $n \to \infty$ or sometimes simply we write $a_n \to a$.

2. The inequality
$$|a_n - a| < \epsilon \;\; \forall \;\; n \geq k$$
is also written as
$$a - \epsilon < a_n < a + \epsilon \;\; \forall \;\; n \geq k$$
or
$$a_n \in (a - \epsilon, a + \epsilon) \;\; \forall \;\; n \in k$$
Thus, $\lim\limits_{n\to\infty} a_n = a$, if and only if for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that
$$a_n \in (a - \epsilon, a + \epsilon) \;\; \forall \;\; n \geq k.$$

3. Suppose $< a_n >$ is a sequence and $a \in \mathbb{R}$. Then to show that $< a_n >$ does not converge to a, we should be able to find an $\epsilon > 0$ such that infinitely may terms of the sequence are outside the interval $(a - \epsilon, a + \epsilon)$ or there exist $k \in \mathbb{N}$, such that $a_n \notin (a - \epsilon, a + \epsilon) \;\; \forall \;\; n \geq k$.

4. The different values of ϵ can result in different N, i.e. the number N may vary as ϵ varies.

Example 7.12. Prove that every constant sequence is a convergent sequence.

Solution. Let $< a_n >=< c >$ be a constant sequence, where $c \in \mathbb{R}$. Then, for any given $\epsilon > 0$, there exists positive integer $k = 1 \in \mathbb{N}$

$$|a_n - c| = |c - c| = 0 < \epsilon \;\; \forall \;\; n \geq k = 1. \tag{7.7}$$

Therefore, by the definition a_n converges to c. Thus, the given constant sequence is convergent and converges to the constant term of the sequence.

Example 7.13. Show that the sequence $< \frac{1}{n} >$ is convergent and it converges to 0.

Solution. Since the given sequence is $< a_n >=< \frac{1}{n} >=< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... >$. If the given sequence is convergent then according to definition of convergence, for every $\epsilon > 0$, there exists positive integer N (in general depending on ϵ) such that $|a_n - l| < \epsilon, \quad \text{for} \quad n \geq N$. Let ϵ be an arbitrary positive real number. Then

$$|a_n - 0| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon, \quad \text{for} \quad n > \frac{1}{\epsilon} \tag{7.8}$$

Let k be a positive integer such that $k > \frac{1}{\epsilon}$. Then,

$$|a_n - 0| < \epsilon, \quad \text{for} \quad n \geq k,$$

Hence, the given sequence is convergent and converges to 0. That is $\lim\limits_{n \to \infty} \frac{1}{n} = 0$.

7.4.3 Non-Convergent Sequences

A sequence which does not converge is called a divergent sequence.

Definition 7.11. If a sequence $< a_n >$ is such that for every $M > 0$, there exists $k \in \mathbb{N}$ such that

$$a_n > M, \quad \forall \quad n \geq k,$$

then we say that $< a_n >$ diverges to $+\infty$ and it is denoted as $\lim\limits_{n \to \infty} a_n = +\infty$.

Definition 7.12. If a sequence $< a_n >$ is such that for every $M > 0$, there exists $k \in \mathbb{N}$ such that

$$a_n < -M, \quad \forall \quad n \geq k,$$

then we say that $< a_n >$ diverges to $-\infty$ and it is denoted as $\lim\limits_{n \to \infty} a_n = -\infty$.

Definition 7.13. A sequence that diverges to either $+\infty$ or $-\infty$ is said to be a divergent sequence.

Definition 7.14. A sequence that diverges to neither $+\infty$ nor $-\infty$ is said to be a non divergent sequence.

Example 7.14. Following sequences are non convergent sequences:

1. The sequence $< n^2 >=< 1, 4, 9, 16, ... >$ diverges to $+\infty$.

2. The sequence $< -4n >=< -4, -8, -12, -16, ... >$ diverges to $-\infty$.

3. The sequence $< (-1)^n.n >=< -1, 2, -3, 4, ... >$ neither diverges to $+\infty$ nor $-\infty$.

Definition 7.15. A bounded sequence is said to **oscillate finitely**, if it is neither convergent nor divergent.

Definition 7.16. A sequence is said to **oscillate infinitely**, if

1. it is not bounded and

2. it neither converges nor diverges.

Example 7.15.

The sequence $< (-1)^n > = < -1, 1, -1, 1, ... >$ oscillates finitely.

The sequence $< (-1)^n.n > = < -1, 2, -3, 4, -5, 6, ... >$ oscillates infinitely.

The sequence $< 1 + (-1)^n > = < 0, 2, 0, 2... >$ oscillates finitely.

Definition 7.17 (Cauchy Sequence). A sequence $< a_n >$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \ \ \forall \ \ n, m \geq k.$$

Theorem 7.1. *If* $< a_n >$, $< b_n >$ *be two convergent sequences such that* $\lim a_n = a$, $\lim b_n = b$, *then*

(i) $\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n = a \pm b.$

(ii) $\lim(a_n b_n) = (\lim a_n) \cdot (\lim b_n) = ab.$

(iii) $\lim \left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n} = \frac{a}{b}. \quad (b \neq 0, \ b_n \neq 0 \ \forall n)$

Example 7.16. Prove that $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0.$

Solution. We know, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Therefore by theorem 7.1, we have

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 0.$$

Theorem 7.2 (Sandwich (Squeeze) Theorem)**.** *Let* f, g *and* h *be real functions such that* $f(x) \leq g(x) \leq h(x)$ *for all* x *in the common domain of definition. For some real number* a*, if*

$$\lim_{x\to a} f(x) = l, \ \lim_{x\to a} h(x) = l,$$

then

$$\lim_{x\to a} g(x) = l.$$

Example 7.17. Using the above inequality prove that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Solution. We know,

$$\cos x < \frac{\sin x}{x} < 1.$$

Also, it is clear that $\lim\limits_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$. Hence, by squeeze theorem 7.2,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Hence, we proved.

Theorem 7.3 (D'Alemberts Limit Theorem)**.** *Let* $< a_n >$ *be a sequence of positive real numbers such that* $L = \lim \left(\frac{a_{n+1}}{a_n} \right)$ *exists. If* $L < 1$*, then* $< a_n >$ *converges and* $\lim(a_n) = 0$.

Example 7.18. Prove that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Solution. Let $< a_n > = < \frac{x^n}{n!} >$ be a sequence of real numbers. Then

$$a_n = \frac{x^n}{n!}, \quad a_{n+1} = \frac{x^{(n+1)}}{(n+1)!}$$

and

$$\frac{a_{n+1}}{a_n} = \frac{x^{(n+1)}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}. \tag{7.9}$$

Also,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1} = \frac{\lim\limits_{n\to\infty} \frac{x}{n}}{1 + \lim\limits_{n\to\infty} \frac{1}{n}} = 0 < 1.$$

Hence, by the above theorem 7.3

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Remark. Some important results based on the theorems of sequences are listed below

1. A sequence cannot converge to more than one limit.
2. Every convergent sequence is bounded but the converse is not true.
3. A sequence of real numbers converges if and only if it is a Cauchy Sequence.
4. If $\lim\limits_{n\to\infty} a_n = l$, then $\lim\limits_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = l$.

In-text Exercise 7.4. Solve the following questions:

1. By using the definition of convergence, show that
$$\lim_{n\to\infty} \frac{1 + 2 + 3... + n}{n^2} = \frac{1}{2}$$
2. Show that the sequence $< (-3)^n >$ does not converges.
3. Prove that $\lim\limits_{n\to\infty} \frac{2n^2 + 3}{3n^2 + 5n} = \frac{2}{3}$
4. If $a_n = 2 - \frac{1}{2^n}$, find $\lim\limits_{n\to\infty} a_n$.
5. Prove that $\lim\limits_{x\to 0} \frac{1 - \cos x}{x} = 0$.

7.5 Infinite Series

An infinite series of real numbers is the sum of infinitely many terms of a sequence of real numbers and it is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

where each a_n is a real number. a_n is called the nth term of the series $\sum_{n=1}^{\infty} a_n$.

Example 7.19. Following are the examples of series of real numbers:

1. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

Now the next question arise, how we can find the sum of infinite series. Because adding an infinite number of terms in not a easy task. Therefore, instead of finding the sum of infinite terms we will find the limit of its **sequence of partial sums**. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + ... + a_k$$

Therefore, the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is the sequence $< S_n >$, where

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + ... + a_n. \end{aligned}$$

Let's take an example to see how partial sums can be used to evaluate infinite series.

Example 7.20. Suppose water is flowing from a tank into a pond such that 1000 liters enters the pond in the first hour. During the second hour, an additional 500 liters of water enters the pond. The third hour, 250 liters more water enters into the pond. Assume this pattern continues such that each hour half as much water enters the pond as did the previous hours. If this continues forever, what can we say about the amount of water in the pond? Will the amount of water continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of water in the pond after k hours. Let S_k denote the amount of water in the pond (measured in thousands of liters) after k hours, we see that

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 0.5 = 1 + \frac{1}{2} \\
S_3 &= 1 + 0.5 + 0.25 = 1 + \frac{1}{2} + \frac{1}{4} \\
S_4 &= 1 + 0.5 + 0.25 + 0.125 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\
S_5 &= 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \\
&\vdots
\end{aligned}$$

From the above we can observe that the obtained sums follows a pattern, therefore we can find the amount of water in the pond after k hours as

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

Thus, we have the sequence of partial sums as $< S_k >=< S_1, S_2, ..., S_k, ... >$. Now, we wants to find what happens as $k \to \infty$. Symbolically, the amount of water in the pond as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$$

At the same time, as $k \to \infty$, the amount of water in the pond can be calculated by evaluating $\lim_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums S_k . If the sequence of partial sums $< S_k >$ converges, we say that the infinite series converges, and its sum is given by $\lim_{k\to\infty} S_k$. If the sequence $< S_k >$ diverges, we say the infinite series diverges. Now we will determine the limit of the sequence of partial sums $< S_k >$. By, simplifying some of the obtained partial sums, we see that

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_3 &= 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} \\
S_5 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16}.
\end{aligned}$$

Plotting some of these values, it appears that the sequence $< S_k >$ could be approaching 2.

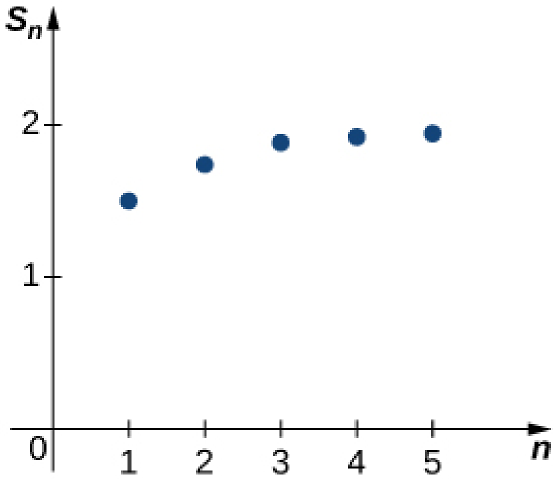


Figure 7.6: The graph shows that the sequence of partial sums $< S_k > \to 2$ as $n \to \infty$.

Since, this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Thus, we conclude that the amount of water in the pond will get arbitrarily close to 2000 liters as the amount of time gets sufficiently large.
This series is an example of a geometric series. We will provide an analytic way later that can be used to prove that $\lim_{k\to\infty} S_k = 2$.

7.6 Convergence and Divergence of an Infinite Series

Consider the infinite series $\sum_{n=1}^{\infty} a_n$. Let $< S_n >$ be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$. Then,

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
\vdots &\quad \vdots \\
S_k &= a_1 + a_2 + a_3 + ... + a_k. \\
\vdots &\quad \vdots
\end{aligned}$$

Definition 7.18. The infinite series $\sum_{n=1}^{\infty} a_n$ is said to converge to the sum S if and only if its sequence of partial sums $< S_n >$ converges to S. Then we write

$$\sum_{n=1}^{\infty} a_n = S.$$

Also, if the sequence of partial sums diverges, we have the divergence of the series, and if the sequence of partial sums oscillates, then the series also oscillates.

Geometric series

A geometric series

$$\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + r^3 + \ldots (r > 0)$$

is the sum of an infinite number of terms that have a constant ratio between successive terms. This geometric series converges if $r < 1$ and it diverges if $r \geq 1$.

Example 7.21. Check the convergence and divergence of the following geometric series:

(a) $\sum \frac{1}{4^n} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$ is convergent. $(\because r = \frac{1}{4} < 1)$

(b) The series $\sum 1 = 1 + 1 + 1 + \ldots$ is divergent. $(\because r = 1)$

(c) $\sum 4^n = 4 + 4^2 + 4^3 + \ldots$is divergent. $(\because r = 4 > 1)$

Example 7.22. Show that the series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots$$

is convergent.

Solution. Let $< S_n >$ be the sequence of partial sums of the given series. Then

$$\begin{aligned} S_n &= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

$$\therefore \lim_{n\to\infty} < S_n > = \lim_{n\to\infty} \left\langle 1 - \frac{1}{n+1} \right\rangle = 1 - 0 = 1, \text{ as } \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since, the sequence of partial sums $< S_n >$ converges to 1, therefore the given series also converges to 1.

p-series

The infinite series of real numbers

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p \in \mathbb{R},$$

is known as a p-series. It converges if $p > 1$ and diverges if $p \leq 1$.

Example 7.23. Check the convergence or divergence of the following p-series:

1. $\sum \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ converges. $(\because p = 3 > 1)$
2. $\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges. $(\because p = 1)$
3. $\sum \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$ diverges. $(\because p = \frac{1}{2} < 1)$
4. $\sum \frac{1}{n^{5/2}}$ is convergent. $(\because p = \frac{5}{2} > 1)$

Theorem 7.4 (**A necessary condition for convergence**). *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $< S_n >$ be the sequence of partial sums of the series $\sum a_n$.

Then

$$S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n,$$

$$S_{n-1} = a_1 + a_2 + \ldots + a_{n-1}.$$

Now

$$S_n - S_{n-1} = a_n. \tag{7.10}$$

Since, the series $\sum a_n$ converges, therefore $< S_n >$ converges. Let $\lim_{n\to\infty} S_n = l$, then

$$\lim_{n\to\infty} S_{n-1} = l \tag{7.11}$$

From equation (7.10) and (7.11), we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = l - l = 0.$$

Hence, $\lim_{n\to\infty} a_n = 0$. □

Example 7.24. Show, by an example, that the converse of above theorem is not true i.e. if $\lim_{n\to\infty} a_n = 0$ then series may or may not be convergent.

Solution. The series $\sum a_n = \sum \frac{1}{n}$ is divergent (by p-series).

But $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

Thus, we see that $\lim_{n\to\infty} a_n = 0$, but the series $\sum a_n = \sum \frac{1}{n}$ is divergent.

Remark. If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ cannot converge.

Proof. Suppose $\sum a_n$ converges. Then by the above theorem, $\lim_{n\to\infty} a_n = 0$, which is contrary to the given condition. Hence the remark. □

Example 7.25. Show that series

$$\sum \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

is not convergent

Solution. We have $a_n = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0.$

Hence, by the above remark, the given series is not convergent.

Theorem 7.5 (Cauchy's Principle of Convergence). *A necessary and sufficient condition for a series $\sum a_n$ to converge is that to each $\varepsilon > 0$, there exists a positive integer m, such that*

$$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon, \quad \textit{for all } n \geq m.$$

Example 7.26. Show that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}$$

is not convergent.

Solution. Let $a_n = \left(\frac{1}{n}\right)^{\frac{1}{n}}$, so that $\log a_n = \frac{1}{n} \log \frac{1}{n}$.
or

$$\log a_n = \frac{\log 1 - \log n}{n} = \frac{-\log n}{n}$$

∴

$$\lim_{n\to\infty} \log a_n = -\lim_{n\to\infty} \frac{\log n}{n}, \text{which is } \frac{\infty}{\infty} \text{ form}$$

$$= -\lim_{n\to\infty} \frac{\frac{1}{n}}{1} \quad \text{(by L'Hospital's Rule)}$$

$$= -\lim_{n\to\infty} \frac{1}{n} = 0.$$

Now,

$$\lim_{n\to\infty} \log a_n = 0 \Rightarrow \log\left(\lim_{n\to\infty} a_n\right) = 0$$
$$\Rightarrow \lim_{n\to\infty} a_n = e^0 = 1 \neq 0.$$

Hence $\sum a_n$ is not convergent.

In-text Exercise 7.5. Solve the following questions:

1. Test for convergence of the series $\sum \cos\left(\frac{1}{n}\right)$.

2. Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}$ oscillates.

3. Show that the series
$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + ... + \sqrt{\frac{n}{2(n+1)}} + ...$$
is not convergent.

4. Test for the convergence of the series $\sum_{n=1}^{\infty}(-1)^n \cdot n$.

7.7 Summary

Following points have been discussed in this lesson:

1. Maxima and minima are the peaks and valleys in the curve of a function.

2. There can be only one absolute maxima of a function and one absolute minimum of a function over the entire domain, whereas there may be several local maxima and local minima.

3. The maxima and minima are collectively known as the "Extrema".

4. If there is a function that is continuous, it must have maxima and minima or local extrema. Also, if the given function is monotonic, the maximum and minimum values lie at the endpoints of the domain of the definition of that function.

5. The concept of Maxima and Minima is used to solve some practical problems.

6. A sequence $< a_n >$ in $\mathbb{R}$ is said to converge to a real number a if for every $\epsilon > 0$, there exists positive integer N (in general depending on ϵ) such that
$$|a_n - a| < \epsilon, \ \ \forall \ \ n \geq N,$$
where the number a is called the limit of the sequence.

7. Geometric series $\sum_{n=1}^{\infty} r^{n-1}$ converges for $|r| < 1$ and diverges for $|r| \geq 1$.

8. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

9. If the series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

7.8 Self-Assessment Exercises

1. Find the local maxima and local minima of the function $f(x) = \sin^4 x + \cos^4 x, \ \ 0 < x < \frac{\pi}{2}$ using the first derivative test.

2. Find the point of local maxima and local minima of the function $f(x) = x\sqrt{1-x}, \ \ x > 0$ using the first derivative test. Also, find the local maximum and local minimum values.

3. Show that $\sin^p \theta \cos^q \theta$ attains a maximum, when $\theta = \tan^{-1} \sqrt{\frac{p}{q}}$.

4. Show that $\frac{\log x}{x}$ has a maximum value at $x = e$.

5. Find all the points of local maxima and local minima and the corresponding maximum and minimum values of the function
$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$$

6. Show that minimum value of the function $f(x) = x^{50} - x^{20}$ in the interval $[0, 1]$ is $-\frac{3}{5}\left(\frac{2}{5}\right)^{\frac{2}{3}}$.

7. Find the point of local maxima and local minima, if any for the given function
$$f(x) = \sin x + \frac{1}{2}\cos 2x, \text{ where } 0 \leq x \leq \frac{\pi}{2}.$$

8. Find the maximum and the minimum value of the function
$$f(x) = 3x^4–8x^3 + 12x^2–48x + 25$$
on the closed interval $[0, 3]$.

9. Find the global maxima and global minima of the function $f(x) = \sin x + \sin x \cos x$, in the interval $[0, \pi]$.

10. Show that of all the rectangles of given area, the square has the smallest perimeter.

11. Show that of all the rectangles inscribed in a given circle, the square has the maximum area.

12. Show that $\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\left(\frac{1}{n}\right)} = 1$ and $\lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1$.

13. Show that $< 1 + (-1)^n >$ is not convergent.

14. By using the definition of convergence, show that

$$\lim_{n\to\infty}\frac{1+3+5+...+2n-1}{n^2} = 1.$$

15. Show that the series $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}$ diverges.

16. Test for the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)^{\frac{1}{n}}$.

7.9 Solutions to In-text Exercises

Exercise 7.1

1. The point $x = 2$ is neither a point of local maxima nor minima. It is a point of inflexion.

2. $f(x)$ attains a local maximum at $x = \frac{\pi}{4}$.

3. None in the interval $(0, \pi)$

4. $x = 0$ is a point of local minima and minimum value is $f(x) = -4$. $x = -2$ is a point of local maxima and maximum value is $f(x) = 0$. Also, there is no point of inflexion.

Exercise 7.2

1. $x = 1$ is a point of local maxima and the maximum value is -3,
   $x = 6$ is a point of local minima and the local minimum value is -128.

2. Since $f'(x) = 0$ does not have any real root, therefore $f(x)$ does not have a maximum or minimum.

3. (i) Local maximum at $x = -2$ and the local maximum value is 0,
   Local minima at $x = 0$ and local minimum value is -4.

   (ii) Local maximum at $x = \frac{3}{4}$ and the local maximum value is $\frac{5}{4}$.

   (iii) Local maximum at $x = \frac{\pi}{4}$ and the local maximum value is $\sqrt{2}$.

(iv) Local maximum at $x = \dfrac{\pi}{6}$ and the local maximum value is $\sqrt{3} + \dfrac{\pi}{6}$,
Local minima at $x = \dfrac{5\pi}{6}$ and the local minimum value is $-\sqrt{3} + \dfrac{5\pi}{6}$.

4. Profit is maximum when $x = \dfrac{2}{3}$ and the maximum value of profit is 49.

5. $a = 2, b = -\dfrac{1}{2}$.

Exercise 7.3

1. The maximum value of $f(x)$ is 0 which is attained at $x = \pi$ and $x = 2\pi$, and the minimum value is -1 which is attained at $x = \dfrac{3\pi}{2}$.

2. Absolute maximum value = 19 at $x = -3$, Absolute minimum value= 3 at $x = 1$.

3. Absolute maximum value = 56 at $x = 5$, Absolute minimum = 24 at $x = 1$.

4. The maximum value of $f(x)$ is 2π and the minimum value is 0.

Exercise 7.4

1. $\left|a_n - \dfrac{1}{2}\right| = \dfrac{1}{2n} < \epsilon$ for $2n > \dfrac{1}{\epsilon}$.

3. $\displaystyle\lim_{n\to\infty} \frac{2n^2+3}{3n^2+5n} = \frac{2 + 3\displaystyle\lim_{n\to\infty}\left(\frac{1}{n^2}\right)}{3 + 5\displaystyle\lim_{n\to\infty}\left(\frac{1}{n}\right)} = \frac{2}{3}$.

4. $\displaystyle\lim_{n\to\infty} a_n = 2$.

Exercise 7.5

1. Since $\displaystyle\lim_{n\to\infty} a_n \neq 0$. Hence the given series is not convergent.

2. Since the sequence of partial sums oscillates, hence the series oscillates.

3. Since $\displaystyle\lim_{n\to\infty} a_n = \frac{1}{\sqrt{2}}$. Hence the given series is not convergent.

4. Sine the sequence of partial sums is unbounded, hence not convergent. Thus the series is not convergent.

7.10 Suggested Readings

1. Narayan, Shanti (Revised by Mittal, P. K.). Differential Calculus. S. Chand, Delhi, 2019.

2. Prasad, Gorakh (2016). Differential Calculus (19th ed.) Pothishala Pvt. Ltd. Allahabad.

3. Thomas Jr., George B., Weir, Maurice D.,Hass, Joel (2014). Thomas Calculus (13th ed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 8

Indeterminate Forms

Structure

8.1 Learning Objectives

The learning objectives of this lesson are to:

- understand the various types of indeterminate forms and conditions to their formation.
- learn the L'Hôpital's Rule.
- use L'Hôpital's Rule to evaluate indeterminate forms arising from limits of products, differences, quotient and exponentials.
- apply the methods of indeterminate forms for evaluating various tedious limits.

8.2 Introduction

In Mathematics, the solution to a problem becomes indeterminate when the information available is insufficient to solve the problem. The problem of evaluation of the limit of a function becomes indeterminate if the method discussed in the previous lessons do not work to completely evaluate the limit. in order to evaluate the limit in indeterminate form, we usually use L'Hôpital's Rule. In this lesson, we shall discuss the indeterminate forms of limits with examples. Also, we will learn how to apply L'Hôpital's Rule to evaluate the limits of the several indeterminate forms. In most of the cases, the indeterminate form occurs while taking the ratio of two functions, such that both of the functions approaches zero in the limit. Such cases are called "indeterminate form $\frac{0}{0}$". Similarly, the indeterminate form can be obtained in addition, subtraction, multiplication, exponential operations also.

8.3 Indeterminate Forms of Limits

Some forms of limits are called indeterminate if the limiting behavior of individual parts of the given expression is not able to determine the limit. Indeterminate forms occur in various types. To understand the indeterminate form, it is important to learn about its types. In Calculus, following types of indeterminate forms of limits occur frequently:

$$\frac{0}{0} \text{ form}, \ \frac{\infty}{\infty} \text{ form}, \ \infty - \infty \text{ form}, \ 0 \times \infty \text{ form}, \text{ and forms of the type } 1^{\infty}, \infty^{0}, \text{ and } 0^{0}.$$

For example, on the basis of the information available from the previous lessons, we can not decide on $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ when $\lim\limits_{x\to a} g(x) = 0$. In order to evaluate this limit (if it exists) of $\dfrac{f(x)}{g(x)}$, (when $\lim\limits_{x\to a} g(x) = 0$), we require the additional information that $\lim\limits_{x\to a} f(x)$ should also be 0. On the contrary, let us assume that $\lim\limits_{x\to a} f(x) \neq 0$ and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = l$,(finite). Then

$$f(x) = \frac{f(x)}{g(x)} \cdot g(x), \quad g(x) \neq 0.$$

$$\begin{aligned} \implies \lim_{x\to a} f(x) &= \lim_{x\to a} \frac{f(x)}{g(x)} \cdot \lim_{x\to a} g(x) \\ &= l \cdot 0 = 0, \end{aligned}$$

which is a contradiction to our supposition. Hence, for $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ to exist provided $\lim\limits_{x\to a} g(x) = 0$, we must have $\lim\limits_{x\to a} f(x) = 0$. Thus this form of limit is an indeterminate form of limit. We now discuss these indeterminate forms of limits, separately.

8.4 Indeterminate Form $\frac{0}{0}$

The limit of the type $\lim_{x\to a} \frac{f(x)}{g(x)}$ is said to be an indeterminate form of the type $\frac{0}{0}$, if $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.

It is evaluated by using the L'Hôpital's Rule as following:

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) Derivatives $f'(x)$ and $g'(x)$ exist and $g'(x) \neq 0 \; \forall x \in [a-\delta, a+\delta] \;\; \delta > 0$.

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Remark. Some basic remarks on L'Hôpital's Rule

1. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist and again it is an indeterminate form of the type $\frac{0}{0}$, then the Rule is repeated again i.e. $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$, and hence

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}.$$

2. This Rule can be extended to any order, That is, if $\lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ is of the indeterminate form $\frac{0}{0}$, then

$$\lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x\to a} \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)},$$

and hence

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \ldots = \lim_{x\to a} \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}.$$

Example 8.1. Evaluate

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

Solution. Let $f(x) = \sin x$ and $g(x) = x$. Then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin x = 0$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0.$$

Therefore, $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{\sin x}{x}$ is an indeterminate form of the type $\dfrac{0}{0}$. Therefore, by using the L'Hôpital's Rule, we have

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

$$\text{i.e.} \quad \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example 8.2. Evaluate

$$\lim_{x\to 0} \frac{\log(1-x^2)}{\log(\cos x)}$$

Solution. We have $f(x) = \log(1-x^2)$ and $g(x) = \log(\cos x)$,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \log(1-x^2) = \log[\lim_{x\to 0}(1-x^2)] = \log 1 = 0.$$

and

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \log(\cos x) = \log 1 = 0.$$

Hence, the given limit is of the indeterminate form $\dfrac{0}{0}$. Therefore, using the L'Hôpital's Rule

$$\begin{aligned}
\lim_{x\to 0} \frac{\log(1-x^2)}{\log(\cos x)} &= \lim_{x\to 0} \frac{\frac{d}{dx}\log(1-x^2)}{\frac{d}{dx}\log(\cos x)} \\
&= \lim_{x\to 0} \frac{\left(\frac{-2x}{1-x^2}\right)}{\left(\frac{-\sin x}{\cos x}\right)} \\
&= \lim_{x\to 0} \frac{\left(\frac{-2x}{1-x^2}\right)}{(-\tan x)} \\
&= \lim_{x\to 0} \frac{2x}{(1-x^2)\tan x} \quad \left(\frac{0}{0} \text{ form}\right).
\end{aligned}$$

Since, the above expression is again in the form of $\dfrac{0}{0}$, hence we again apply L' Hospital Rule

$$\therefore \lim_{x\to 0} \frac{2x}{(1-x^2)(\tan x)} = \lim_{x\to 0} \frac{2}{(1-x^2)(\sec x)^2 + \tan x \cdot (-2x)} = \frac{2}{1\cdot 1 + 0} = 2.$$

Thus, finally we have

$$\lim_{x\to 0} \frac{\log(1-x^2)}{\log(\cos x)} = 2.$$

Example 8.3. Find the value of

$$\lim_{x\to 0} \frac{e^x - 1}{x^2 + x}$$

Solution. Let $f(x) = e^x - 1$ and $g(x) = x^2 + x$.

$$\implies \lim_{x\to 0} f(x) = \lim_{x\to 0}(e^x - 1) = 1 - 1 = 0$$

and

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(x^2 + x) = 0$$

Hence, the given limit is of the indeterminate form $\frac{0}{0}$. Therefore, by using the L' Hospital Rule , we get

$$\begin{aligned}\lim_{x\to 0}\frac{e^x - 1}{x^2 + x} &= \lim_{x\to 0}\frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(x^2 + x)}\\ &= \lim_{x\to 0}\frac{e^x}{2x + 1}\\ &= \frac{e^0}{2\cdot 0 + 1}\\ &= 1.\end{aligned}$$

Example 8.4. Find the value of a and b for which

$$\lim_{x\to 0}\frac{x(1 + a\cos x) - b\sin x}{x^3}$$

exists and is equal to 1.

Solution. Since

$$\lim_{x\to 0}\frac{x(1 + a\cos x) - b\sin x}{x^3}$$

is of the indeterminate form $\frac{0}{0}$, therefore by using the L' Hospital Rule , we have

$$\begin{aligned}\lim_{x\to 0}\frac{x(1 + a\cos x) - b\sin x}{x^3} &= \lim_{x\to 0}\frac{\frac{d}{dx}(x(1 + a\cos x) - b\sin x)}{\frac{d}{dx}(x^3)}\\ &= \lim_{x\to 0}\frac{1 + a\cos x - a\sin x - b\cos x}{3x^2}\\ &= \frac{1 + a - b}{0}. \qquad (8.1)\end{aligned}$$

Since, it is given that

$$\lim_{x\to 0}\frac{x(1 + a\cos x) - b\sin x}{x^3}$$

is finite and is 1. Therefore, to get the finite limit, we should have

$$1 + a - b = 0. \qquad (8.2)$$

Then the limit in 8.1 is again in $\frac{0}{0}$ form

$$i.e.\ \lim_{x\to 0}\frac{1 + a\cos x - a\sin x - b\cos x}{3x^2} \quad \text{is} \left(\frac{0}{0}\text{ form}\right).$$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{-a\sin x - a\sin x - ax\cos x + b\sin x}{6x} \quad \left(\frac{0}{0} \text{ form again,}\right) \text{ by L'Hôpital's Rule} \\
&= \lim_{x\to 0} \frac{-2a\cos x + ax\sin x - a\cos x + b\cos x}{6} \\
&= \frac{-2a - a + b}{6} \\
&= \frac{-3a + b}{6}. \text{ by the L'Hospital's Rule.}
\end{aligned}
$$

Since, the given limit is 1

$$
\therefore \frac{-3a+b}{6} = 1
$$

$$
-3a + b = 6. \tag{8.3}
$$

On Solving 8.2 and 8.3 for a and b, we get

$$
a = -\frac{5}{2} \text{ and } b = -\frac{3}{2}.
$$

8.5 Indeterminate Form $\frac{\infty}{\infty}$

A limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ is said to be an indeterminate form of the type $\frac{\infty}{\infty}$, if $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$.

It is evaluated by using the L'Hôpital's Rule as following:

If

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$.

(ii) Derivatives $f'(x)$ and $g'(x)$ exist and $g'(x) \neq 0$ for $x \in (a-\delta, a+\delta)$ $\delta > 0$.

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then,

$$
\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.
$$

Remark. On applying the L'Hôpital's Rule once, if the resulting limit of quotient is again is an indeterminate form, then we apply the L'Hôpital's Rule repeatedly until we get a finite limit of the quotient.

Example 8.5. Compute the following limit

$$
\lim_{x\to\infty} \frac{2x^2+3}{5x^2+x}.
$$

Solution. Since

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2+x}$$

is an indeterminate form of type $\dfrac{\infty}{\infty}$. Therefore, using L'Hôpital's Rule, we get

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2+x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(2x^2+3)}{\frac{d}{dx}(5x^2+x)} = \lim_{x\to\infty} \frac{4x}{10x+1} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$
$$= \frac{4}{10} = \frac{2}{5}.$$

Example 8.6. Evaluate

$$\lim_{x\to\infty} \frac{e^x}{x^2}.$$

Solution. Here,

$$\lim_{x\to\infty} e^x = \infty = \lim_{x\to\infty} x^2.$$

Therefore, $\lim_{x\to\infty} \dfrac{e^x}{x^2}$ is of $\dfrac{\infty}{\infty}$ indeterminate form. Hence, by applying the L'Hôpital's Rule, we get

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} \quad \left(\frac{\infty}{\infty} \text{ form}\right),$$
$$= \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Example 8.7. Evaluate

$$\lim_{x\to a} \frac{\log(x-a)}{\log(e^x-e^a)}.$$

Solution. Since

$$\lim_{x\to a} \frac{\log(x-a)}{\log(e^x-e^a)} \text{ is of } \frac{\infty}{\infty} \text{ indeterminate form,}$$

therefore, using L'Hôpital's Rule, we get

$$\lim_{x\to a} \frac{\log(x-a)}{\log(e^x-e^a)} = \lim_{x\to a} \frac{\frac{1}{(x-a)}}{\frac{e^x}{e^x-e^a}}$$
$$= \lim_{x\to a} \frac{e^x-e^a}{e^x(x-a)} \text{ is of } \left(\frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to a} \frac{e^x}{e^x(x-a)+e^x}$$
$$= \frac{e^a}{e^a}$$
$$= 1$$

In-text Exercise 8.1. Solve the following questions:

1. If
$$\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$$
is finite, find the value of a and the limit.

2. Evaluate
$$\lim_{x\to 0}\frac{\tan x - x}{x^2\tan x}.$$

3. Prove that
$$\lim_{x\to 0}\frac{\tan x}{x} = 1.$$

4. Evaluate
$$\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}.$$

5. Evaluate
$$\lim_{x\to 1}\frac{\log(x-1)+\tan\frac{\pi x}{2}}{\cot \pi x}$$

6. Find the value of
$$\lim_{x\to 0}\frac{e^x\sin x - x - x^2}{x^3}$$

8.6 Indeterminate Form $\infty - \infty$

If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a}(f(x)-g(x))$ is said to be of the indeterminate form $\infty - \infty$.

This type of limit is evaluated by rearranging the terms and converting it into the $\frac{0}{0}$ indeterminate form. We write

$$\begin{aligned}\lim_{x\to a}(f(x)-g(x)) &= \lim_{x\to a}\left(\frac{f(x)-g(x)}{f(x)g(x)}\right) f(x)g(x)\\ &= \lim_{x\to a}\frac{\left(\frac{1}{g(x)}-\frac{1}{f(x)}\right)}{\frac{1}{f(x)g(x)}}\end{aligned}$$

which is of the form $\frac{0}{0}$ as $\lim\limits_{x\to a} f(x) = \infty \implies \lim\limits_{x\to a}\frac{1}{f(x)} = 0$ and $\lim\limits_{x\to a} g(x) = \infty \implies \lim\limits_{x\to a}\frac{1}{g(x)} = 0$. The resulting $\frac{0}{0}$ form is then evaluated by using L'Hôpital's Rule.

Example 8.8. Find the value of

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$$

Solution. Since

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) \quad \text{is of} \quad \infty-\infty \quad \text{form},$$

therefore, it can be rearranged as

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) &= \lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right) \quad \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) \quad \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\left(\frac{-\sin x}{\cos x - x\sin x + \cos x}\right)\\
&= \lim_{x\to 0}\left(\frac{-\sin x}{-x\sin x + 2\cos x}\right)\\
&= 0
\end{aligned}$$

Alternately,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) &= \lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right)\\
&= \lim_{x\to 0}\left(\frac{\sin x - x}{x^2}\right)\cdot \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\\
&= \lim_{x\to 0}\frac{\sin x - x}{x^2} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]\\
&= \lim_{x\to 0}\frac{\cos x - 1}{2x} \quad \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{-\sin x}{2}\\
&= 0.
\end{aligned}$$

8.7 Indeterminate Form $0 \times \infty$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then $\lim_{x\to a}(f(x)\cdot g(x))$ is said to be a limit of the indeterminate form $0\times\infty$.

To evaluate this type of limit it is converted to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form first and then evaluated.

$$\begin{aligned}
\text{We write} \quad \lim_{x\to a} f(x)\cdot g(x) &= \lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}} \quad \left(\frac{0}{0}\text{ form}\right)\\
\text{or} \quad &= \lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)
\end{aligned}$$

which can be solved by using L'Hôpital's Rule.

Example 8.9. Evaluate

$$\lim_{x\to\infty} x^2 \cdot \sin\left(\frac{1}{x^2}\right)$$

Solution. Since, $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} \sin\left(\frac{1}{x^2}\right) = 0$, therefore $\lim_{x\to\infty} x^2 \cdot \sin\left(\frac{1}{x^2}\right)$ is an indeterminate form of type $0 \times \infty$. We write it as

$$\begin{aligned}
\lim_{x\to\infty} x^2 \cdot \sin\left(\frac{1}{x^2}\right) &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} \quad \text{which is of the form } \frac{0}{0}.\\
&= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x^2}\right)\cdot\left(\frac{-2}{x^3}\right)}{\frac{-2}{x^3}}, \text{ by using L'Hospital's Rule}\\
&= \lim_{x\to\infty} \cos\left(\frac{1}{x^2}\right)\\
&= 1.
\end{aligned}$$

Example 8.10. Find the limit

$$\lim_{x\to\infty} (x+6)\cdot\left(\frac{1}{x^2+3}\right)$$

Solution. Since,

$$\lim_{x\to\infty} (x+6)\cdot\left(\frac{1}{x^2+3}\right)$$

is of $\infty \cdot 0$ form, therefore it can be written as

$$\begin{aligned}
\lim_{x\to\infty} (x+6)\cdot\left(\frac{1}{x^2+3}\right) &= \lim_{x\to\infty} \frac{(x+6)}{(x^2+3)} \quad \left(\frac{\infty}{\infty} \text{ form}\right)\\
&= \lim_{x\to\infty} \frac{1}{2x}, \text{ by using L'Hospital's Rule}\\
&= 0.
\end{aligned}$$

In-text Exercise 8.2. Solve the following questions:

1. Prove that $\lim_{x\to0+} x^m(\log x)^n;\quad m.n \in \mathbb{N}$ is zero.

2. Evaluate the value of
$$\lim_{x\to0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right).$$

3. Solve
$$\lim_{x\to0}\left(\frac{1}{x^2} - \frac{\csc x}{x}\right) \qquad \csc x = cosecx.$$

4. Evaluate
$$\lim_{x\to4}\left[\frac{1}{\log(x-3)} - \frac{1}{x-4}\right].$$

5. Evaluate
$$\lim_{x\to\infty}\left(x\tan\frac{1}{x}\right).$$

8.8 Indeterminate Forms $1^\infty, \infty^0$ and 0^0

Consider $\lim\limits_{x\to a}(f(x))^{g(x)}$. It is a indeterminate form of the type

(i) 1^∞, if $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$.

(ii) ∞^0, if $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = 0$.

(iii) 0^0, if $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$.

To evaluate these types of limits, we write

$$y = (f(x))^{g(x)}.$$

Taking $\log$ on both sides, we have

$$\begin{aligned} \log y &= g(x)\log(f(x)) \\ \implies \lim_{x\to a}\log y &= \lim_{x\to a}[g(x)\log(f(x))]. \end{aligned} \tag{8.4}$$

Now, the limit on the right side of 8.4 can be reduced either in the $\dfrac{\infty}{\infty}$ form or $\dfrac{0}{0}$ form, which can be evaluated by using the L'Hôpital's Rule. Suppose,

$$\lim_{x\to a}(g(x)\log(f(x)) = l$$

then equation 8.4 becomes,

$$\begin{aligned} \lim_{x\to a}\log y &= l \\ \text{or} \quad \log\lim_{x\to a} y &= l \\ \lim_{a\to a} y &= e^l \\ \text{Thus} \quad \lim_{x\to a}(f(x))^{g(x)} &= e^l \quad \text{where } \ l = \lim_{x\to a} g(x)\log(f(x)). \end{aligned}$$

Example 8.11. Find the value of

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{(\tan x)^2}$$

Solution. Since,

$$\begin{aligned} \lim_{x\to\frac{\pi}{2}} f(x) &= \sin\left(\frac{\pi}{2}\right) = 1 \\ \lim_{x\to\frac{\pi}{2}} g(x) &= \left(\tan\left(\frac{\pi}{2}\right)\right)^2 = \infty \end{aligned}$$

Therefore, $\lim_{x\to\frac{\pi}{2}}(\sin x)^{(\tan x)^2}$ is of the form $(1)^\infty$. Let $y = (\sin x)^{(\tan x)^2}$, then taking $\log$ on both side, we get

$$\begin{aligned}
\log y &= (\tan x)^2 \log(\sin x) \\
\Longrightarrow \lim_{x\to\frac{\pi}{2}} \log y &= \lim_{x\to\frac{\pi}{2}} (\tan x)^2 \log(\sin x) \quad (\infty \times 0 \text{ form}) \\
&= \lim_{x\to\frac{\pi}{2}} \frac{\log(\sin x)}{(\cot x)^2} \quad \left(\frac{0}{0} \text{ form}\right) \\
&= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-2\cot x \cdot (\csc x)^2} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{\cot x}{-2\cot x \cdot (\csc x)^2} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{-1}{2(\csc x)^2} = -\frac{1}{2} \\
\lim_{x\to\frac{\pi}{2}} \log y &= \log(\lim_{x\to\frac{\pi}{2}} y) = -\frac{1}{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}} y &= e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}. \\
i.e. \lim_{x\to\frac{\pi}{2}} (\sin x)^{(\tan x)^2} &= \frac{1}{\sqrt{e}}.
\end{aligned}$$

Example 8.12. Evaluate the limit

$$\lim_{x\to 0} x^x.$$

Solution. Since the given limit is of the indeterminate form 0^0. Let

$$\begin{aligned}
y &= x^x \\
\Longrightarrow \log y &= x \log x \\
\Longrightarrow \lim_{x\to 0} \log y &= \lim_{x\to 0} x \log x \quad (0 \times \infty \text{ form}) \\
&= \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&= \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}}, \text{ by using the L'Hospital's Rule} \\
&- \lim_{x\to 0} x = 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\log \lim_{x\to 0} y &= 0 \\
\Longrightarrow \lim_{x\to 0} y &= e^0 \\
\Longrightarrow \lim_{x\to 0} x^x &= \lim_{x\to 0} y = e^0 = 1.
\end{aligned}$$

Example 8.13. Evaluate the following limit

$$\lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x}$$

Solution. Since, the given limit is of the form $(\infty)^0$, therefore let

$$\begin{aligned}
y &= \left(\frac{1}{x}\right)^{1-\cos x}\\
\implies \log y &= (1-\cos x)\log\left(\frac{1}{x}\right)\\
\therefore\ \lim_{x\to 0}\log y &= \lim_{x\to 0}(1-\cos x)\log\left(\frac{1}{x}\right)\\
&= \lim_{x\to 0} 2\left(\sin\frac{x}{2}\right)^2(\log 1-\log x)\\
&= \lim_{x\to 0} 2\left(\sin\frac{x}{2}\right)^2(-\log x)\\
&= \lim_{x\to 0} 2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{x^2}{4}\cdot(-\log x)\\
&= \lim_{x\to 0}\frac{(-\log x\cdot x^2)}{2}\cdot\lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2\\
&= \lim_{x\to 0}\frac{-\log x}{\frac{2}{x^2}}\quad\left[\because\ \lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}=1\right]\left(\frac{\infty}{\infty}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{\frac{-1}{x}}{\frac{-4}{x^3}},\ \text{by using the L'Hospital's Rule}\\
&= \lim_{x\to 0}\left(\frac{x^2}{4}\right)=0\\
\lim_{x\to 0}\log y &= \log\lim_{x\to 0} y = 0\\
\implies \lim_{x\to 0} y &= e^0 = 1.
\end{aligned}$$

Thus

$$\lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x} = \lim_{x\to 0} y = 1.$$

Example 8.14. Evaluate

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{1}{x^2}\right)}$$

Solution. We have

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1\\
\lim_{x\to 0} g(x) &= \lim_{x\to 0}\left(\frac{1}{x^2}\right) = \infty.
\end{aligned}$$

Therefore, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{1}{x^2}\right)}$ is of the indeterminate form $(1)^{\infty}$. Let

$$
\begin{aligned}
y &= \left(\frac{\sin x}{x}\right)^{\left(\frac{1}{x^2}\right)} \\
\implies \log y &= \left(\frac{1}{x^2}\right)\log\left(\frac{\sin x}{x}\right) \\
\lim_{x\to 0}\log y &= \lim_{x\to 0}\left(\frac{1}{x^2}\right)\log\left(\frac{\sin x}{x}\right) \quad (\infty\times 0 \text{ form}) \\
\lim_{x\to 0}\log y &= \lim_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x^2} \quad \left(\frac{0}{0}\text{ form}\right) \\
&= \lim_{x\to 0}\frac{\frac{\frac{x\cos x-\sin x}{x^2}}{\frac{\sin x}{x}}}{2x}, \text{ by using the L'Hospital's Rule.} \\
&= \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x} \\
&= \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3}\cdot\lim_{x\to 0}\left(\frac{x}{\sin x}\right) \\
&= \lim_{x\to 0}\left(\frac{-x\sin x+\cos x-\cos x}{6x^2}\right) \quad \left[\because \lim_{x\to 0}\left(\frac{\sin x}{x}=1\right)\right]. \\
&= \lim_{x\to 0}\frac{-\sin x}{6x} \\
&= \lim_{x\to 0}\frac{-1}{6}\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right) \\
&= \frac{-1}{6} \\
\lim_{x\to 0}\log y &= \log\lim_{x\to 0} y = \frac{-1}{6} \\
\implies \lim_{x\to 0} y &= e^{-\frac{1}{6}}.
\end{aligned}
$$

Hence,

$$\lim_{x\to 0}\log\left(\frac{\sin x}{x}\right)^{\left(\frac{1}{x^2}\right)} = \lim_{x\to 0} y = e^{\frac{-1}{6}}.$$

In-text Exercise 8.3. Evaluate the following limits:

1. $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$.

2. $\lim_{x\to 0}(\csc x)^{\frac{1}{\log x}}$.

3. $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}$.

4. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

5. $\lim_{x\to a}\left(3-\frac{2x^2}{a^2}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$.

8.9 Summary

We have discussed the following points in this lesson:

- Indeterminate forms of limits are of the following forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty, \text{ and } \infty^0$$

- Indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ can be easily evaluated by using L' Hospital Rule which state that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

- The limiting value of an indeterminate form is called the true value of the limit.
- Indeterminate forms $0\times\infty, \infty-\infty, 0^0, 1^\infty,$ and ∞^0 can be easily evaluated by using $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

8.10 Self-Assessment Exercises

1. Evaluate the determinate form:

$$\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}.$$

2. Evaluate the following limits

(a)

$$\lim_{x\to\infty}\left(\frac{\pi}{2}-\tan^{-1}x\right)^{\frac{1}{x}}.$$

(b)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{\log(1+x)}{x^2}\right).$$

(c)

$$\lim_{x\to 0}\frac{1-(\sec x)^2}{3x^2}.$$

(d)

$$\lim_{x\to 0}\frac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}.$$

(e)

$$\lim_{x \to 0} \left(\frac{\sinh x}{x} \right)^{\frac{1}{x}}.$$

3. Show that

$$\lim_{x \to 0} \frac{e^x + \log\left(\frac{1-x}{e}\right)}{\tan x - x} = -\frac{1}{2}.$$

4. Find the value of p, q and r, if

$$\lim_{y \to 0} \frac{re^y - q\cos y + pe^{-y}}{y \tan y} = 3.$$

5. Find the values of a, b, c, so that

$$\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2.$$

6. Show that

$$\lim_{x \to 0+} \frac{(1+x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2} = \frac{11e}{24}.$$

7. Find the value of

$$\lim_{x \to \infty} \frac{2x^4 + 4x^3 - 100}{4x^4 + 9x^2 + 2x + 100}.$$

8. Evaluate

$$\lim_{x \to 0+} \log_x \sin x \qquad \textbf{Hint.} \left[\log_x \sin x = \frac{\log \sin x}{\log x} \right].$$

Solutions to In-text Exercises

Exercise 8.1

1. Considering $a \in \mathbb{R}$ to be finite, the given limit

$$\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$$

is of the form $\frac{0}{0}$, therefore we evaluate it by applying L'Hôpital's Rule to obtain $a = -2$ and the value of limit as -1.

2. The value of the limit is $\frac{1}{3}$.

4. The value of the limit is $\frac{3}{2}$.

5. The value of the limit is $\frac{1}{3}$.

6. The value of the limit is $\frac{1}{3}$.

Exercise 8.2

1. $\lim_{x \to 0+} x^m (\log x)^n; \quad m.n \in \mathbb{N}$ is of the form $(0 \times \infty)$. Therefore, we can convert it either in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hôpital's Rule to evaluate the limit. The value of the limit is 0.

2. $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x} \right]$ is of the form $\infty - \infty$. Therefore, we can convert it either in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hôpital's Rule to evaluate the limit. The value of the limit is $-\frac{1}{3}$.

3. $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{\csc x}{x} \right]$ is of the form $\infty - \infty$. Therefore, we can convert it either in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hôpital's Rule to evaluate the limit. The value of the limit is $-\frac{1}{6}$.

4. $\lim_{x \to 4} \left[\frac{1}{\log(x-3)} - \frac{1}{x-4} \right]$ is of the form $\infty - \infty$. Therefore, we can convert it either in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hôpital's Rule to evaluate the limit. The value of the limit is $\frac{1}{2}$.

5. The value of limit is 1.

Exercise 8.3

1. $\lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$ is of the form $(1)^\infty$ which can be converted either in $\left(\frac{0}{0}\right)$ form or $\left(\frac{\infty}{\infty}\right)$ form. Then use L'hospital's Rule to evaluate the limit. The value of limit is -1.

2. $\lim_{x \to 0} (\csc x)^{\frac{1}{\log x}}$ is of the form $(\infty)^0$. Therefore, by taking $\log$ on both side, we get

$$\begin{aligned}
\lim_{x \to 0} \log y &= \lim_{x \to 0} \frac{1}{\log x} \log(\csc x) \; (0 \times \infty \text{ form}) \\
&= \lim_{x \to 0} \left(\frac{\log \csc x}{\log x} \right) \; \left(\frac{\infty}{\infty} \text{ form} \right) \\
&= -1
\end{aligned}$$

Thus, $\quad \lim_{x \to 0} (\csc x)^{\frac{1}{\log x}} = e^{-1} = \frac{1}{e}$.

3. $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x$ is of the form $(1)^\infty$. Therefore, by taking $\log$ on both side, we get

$$\begin{aligned}\lim_{x\to\infty}\log y &= \lim_{x\to\infty} x\log\left(1+\frac{k}{x}\right) (\infty\times 0 \text{ form})\\ &= \lim_{x\to\infty}\log\left(1+\frac{k}{x}\right)/\frac{1}{x} \quad \left(\frac{0}{0}\text{ form}\right)\\ &= k\end{aligned}$$

Thus, $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = e^k.$

4. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$ is of the form $(1)^\infty$ and the value of limit is e.

5. $\lim_{x\to a}\left(3-\frac{2x^2}{a^2}\right)^{\tan(\frac{\pi x}{2a})}$ is of the form $(1)^\infty$ which can be converted either in $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form. Then use L'hospital's Rule to evaluate the limit. The value of limit is π^2.

8.11 Suggested Readings

1. Narayan, Shanti (Revised by Mittal, P. K.). Differential Calculus. S. Chand, Delhi, 2019.

2. Prasad, Gorakh (2016). Differential Calculus (19th ed.) Pothishala Pvt. Ltd. Allahabad.

3. Thomas Jr., George B., Weir, Maurice D.,Hass, Joel (2014). Thomas Calculus (13th ed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 9

Asymptotes, Concavity and Point of Inflexion

Structure

9.1 Learning Objectives

The learning objectives of this lesson are to:

- understand the notion of asymptotic behaviour of curves
- learn the methods to find the different types of asymptotes
- understand the concept of concavity of a function
- learn to find a point of inflexion.

9.2 Introduction

Informally, an asymptote of a curve is a straight line which touches the curve at infinity. An asymptote indicates the behaviour of a curve at the points far off from the origin. A curve that lies wholly in a bounded region has no asymptote. For example, the circle $x^2+y^2=a^2$ has no asymptote. In this lesson, we will learn about different kinds of asymptotes and the procedures to find them. We will also discuss the important concepts of the concavity and the points of inflexion of a curve. All these concepts are useful in sketching the graphs of various functions. We begin with the formal definition of an asymptote of a curve.

9.3 Asymptote

Definition 9.1. (Asymptote): A straight line is called an asymptote of a given curve $y = f(x)$, if the perpendicular distance between the line and the point $A(x, y)$ on the curve approach to 0 as x or y or both approach to infinity.

There are three kinds of asymptotes, as given below:

- Asymptotes parallel to $x-$ axis or horizontal asymptotes.
- Asymptotes parallel to $y-$ axis or vertical asymptotes.
- Oblique asymptotes.

In the following sub-sections, we will discuss these kinds of asymptotes one by one.

9.3.1 Horizontal Asymptotes

By the definition of an asymptote, a horizontal line $y = c$ is a horizontal asymptote of the curve $y = f(x)$, if the perpendicular distance of the point on the curve from the line $y = c$ tends to 0 as $x \to \infty (or - \infty)$. That is, if $\lim_{x\to\infty} y = c$ or $\lim_{x\to-\infty} y = c$, then $y = c$ is a horizontal asymptote of the curve $y = f(x)$.

Example 9.1. Find the asymptote parallel to $x-$ axis of the curve

$$y = \frac{3x^2 + 8x - 5}{5x^2 + 2}$$

Solution. We have

$$\begin{aligned}\lim_{x\to\infty} y &= \lim_{x\to\infty} f(x)\\ &= \lim_{x\to\infty} \frac{3x^2 + 8x - 5}{5x^2 + 2}\\ &= \lim_{x\to\infty} \frac{3 + 8/x - 5/x^2}{5 + 2/x^2}\\ &= \frac{3}{5}\end{aligned}$$

Therefore the line $y = \frac{3}{5}$ or $5y - 3 = 0$ is a horizontal asymptote to the given curve $y = f(x)$ on the right.
Similarly, we can show that

$$\begin{aligned}
\lim_{x \to -\infty} y &= \lim_{x \to -\infty} f(x) \\
&= \lim_{x \to -\infty} \frac{3x^2 + 8x - 5}{5x^2 + 2} \\
&= \lim_{x \to -\infty} \frac{3 + 8/x - 5/x^2}{5 + 2/x^2} \\
&= \frac{3}{5}
\end{aligned}$$

Therefore, the line $y = 3/5$ is a horizontal asymptote to $y = f(x)$ on the left. That is, the line $y = 5/3$ to a horizontal asymptote or an asymptote parallel to $x-$ axis.

9.3.2 Vertical Asymptotes

A vertical line $x = d$ is said to be a vertical asymptote(asymptote parallel to $y-$ axis) of the curve $y = f(x)$, if the perpendicular distance of the point on the curve from the line $x = d$ tends to 0 as $y \to \infty (or - \infty)$.

Example 9.2. Find the asymptote parallel to $x-$ axis of the curve

$$y = \frac{x^2 + 8x - 3}{x^2 - 5x - 6}$$

Solution. Consider the denominator

$$x^2 - 5x - 6 = (x - 6)(x + 1),$$

which means that two zeros of the denominator are $x = 6$ and $x = -1$.

$$\lim_{x \to 6} \frac{x^2 + 8x - 3}{x^2 - 5x - 6} = \infty$$

$$\lim_{x \to -1} \frac{x^2 + 8x - 3}{x^2 - 5x - 6} = \infty$$

Therefore, the line $x = 6$ and $x = -1$ are vertical asymptote to $y = f(x)$.

Example 9.3. Find the asymptote parallel to $x-$ axis of the curve

$$x^2 + y^2 - a^2(x^2 + y^2) = 0$$

Solution. The given equation can be written as

$$x^2y^2 - a^2x^2 - a^2y^2 = 0$$

$$\Rightarrow (y^2 - a^2)x^2 - a^2y^2 = 0$$

$$\Rightarrow (x^2 - a^2)y^2 - a^2x^2 = 0$$

Clearly, The Asymptotes parallel to x-axis are given by $y^2 - a^2 = 0 \Rightarrow y = \pm a$.

We give the following rule for determining the Asymptotes parallel to the coordinate axes: **RULE 1.** The Asymptotes parallel to x-axis can be easily obtained by equating to zero the coefficient of the highest degree term in x. However if the coefficient of the highest degree term in x is constant or has imaginary(no real) factors then curve has no Asymptote parallel to x-axis.

RULE 2. The Asymptotes parallel to y-axis can be easily obtained by equating to zero the coefficient of the highest degree term in y. However if the coefficient of the highest degree term in y is constant or has imaginary(no real) factors then curve has no Asymptote parallel to x-axis.

9.3.3 Oblique Asymptote

The Asymptote which are not parallel to any of the coordinate Axes are called as *Oblique Asymptote.* An Asymptote of a given curve is a straight line which touches the given curve at infinity

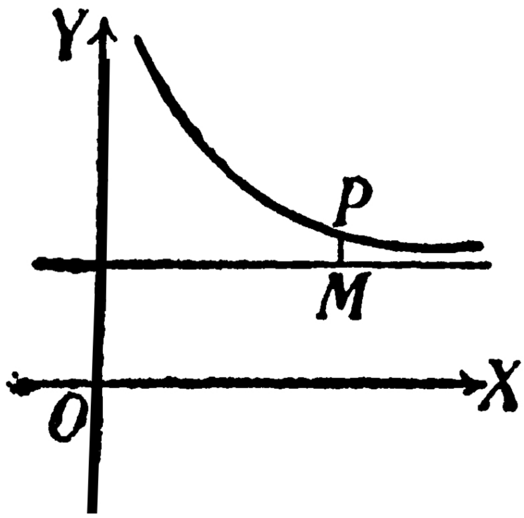


Figure 9.1: Asymptote to a Curve

In other words, a line $y = mx + c$ is called an asymptote of a curve $y = f(x)$ if the perpendicular distance PM of any point P(x,y) on the curve from the line tends to zero as P tends to infinity along the curve.Asymptotes plays a important role in sketching graphs of the function. It gives information about the behaviour of the curve at infinity. In this chapter we will learn about different types of Asymptotes and their finding procedures. In particular we will study about three types of Asymptotes. Firstly Asymptotes parallel to x-axis and y-axis and secondly we will work on finding oblique Asymptotes to the given curve. Concavity and points of inflexion of the curves are also being explained and discussed in this chapter with examples to understand the curves more precisely.

Note:

1. The total number of Asymptotes ,real or imaginary of an algebraic curve of n^{th} degree,

can never exceed n.
2. Non-existence of an Asymptote. If for a curve $y = f(x)$, y/x tend to ∞ as x and y both tends to infinity, then the curve $y - x^3$ has no asymptote.

Definition 9.2. A function f(x,y) in x and y is called as *homogenous function* , if the degree of each term in the expression are equal for e.g. the function

$$f(x,y) = 3x^2 + 2xy + y^2$$

is a *homogenous function* since the sum of the powers of x and y in each term is the same.

Working Rule For Finding Oblique Asymptote
Let the equation of the given curve be written as

$$\phi_n(x,y) + \phi_{n-1}(x,y) + ... + \phi_1(x,y) + c = 0 \qquad (1)$$

where $\phi_r(x,y)$ is homogenous expression in x and y of degree r.

STEP 1. Put $x = 1$ and $y = m$ in (1) to get the polynomials $\phi_n(m)$, $\phi_{n-1}(m)$,$\phi_{n-2}(m)$, etc.

STEP 2. The slopes of the Asymptotes are given by $\phi_n(m)$=0.

STEP 3. If m is non repeated root of $\phi_n(m) = 0$ then the corresponding value of c is given by

$$c = \frac{-\phi_{n-1}(m)}{\phi_n'(m)}$$

STEP 4. If m is repeated root of $\phi_n(m) = 0$ then the two values of 'c' are given by

$$\frac{c^2}{2}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

STEP 5. Similarly in the case when the three roots of equation $\phi_n(m) = 0$ are equal .We get three parallel Asymptotes and the corresponding three values of c are obtained from the equation.

$$\frac{c^3}{3!}\phi_n''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + \frac{c}{1!}\phi_{n-2}(m) + \phi_{n-3}(m) = 0$$

STEP 6. $y = mx + c$ is an Asymptote of the curve.

Example 9.4. Find the Asymptotes of the curve

$$y^3 - x^2y + 2y^2 + 4y + x = 0$$

Solution. The given equation of the curve is of degree 3. We have
$\phi_3(x, y) = y^3 - x^2y$
$\phi_2(x, y) = 2y^2$
$\phi_1(x, y) = 4y + x$

STEP 1. Putting $x = 1$,$y = m$ in the above polynomials ,we have
$\phi_3(m) = m^3 - m = m(m^2 - 1)$
$\phi_2(m) = 2m^2$
$\phi_1(m) = 4m + 1$

STEP 2. Slopes m for the required Asymptotes are given by
$\phi_3(m) = 0$
$\Rightarrow m(m^2 - 1) = 0$
$\Rightarrow m(m - 1)(m + 1) = 0$
$\Rightarrow m = 0, m = 1, m = -1$
which are all distinct and the corresponding c is given by

STEP 3.
$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$
or
$$c = \frac{-2m^2}{3m^2 - 1}$$
Now for m=0 , c=0

m=1, $c = \frac{-2}{2} = -1$

m=-1,$c = \frac{-2}{2} = -1$

Hence the three Asymptotes of the given curve are
$y = 0$ (for $m = 0, c = 0$)
$y = x - 1$ (for $m = 1, c = -1$)
$y = -x - 1$ (for $m = -1, c = -1$)

OBSERVATION

1. In the above question the Asymptote y=0 parallel to x-axis can also be directly obtained by equating to zero the coefficient of the the highest degree term of x which is y=0.

2. Further since the coefficient of the highest degree term of y is constant there is no Asymptote parallel to y axis.

Example 9.5. Find the Asymptotes of the curve

$$xy^2 - 3x^2 - 2xy - x^2y + y^2$$

Solution. We have

$\phi_3(x,y) = xy^2 - x^2y$

$\phi_2(x,y) = -3x^2 - 2xy + y^2$

$\phi_1(x,y) = 0$

STEP 1. Putting x=1 , y=m in the given equation we get,

$\phi_3(m) = m^2 - m$

$\phi_2(m) = -3 - 2m + m^2$

STEP 2. The slopes of the Asymptotes are given by

$\phi_3(m) = 0$ i.e.$m^2 - m = 0$

or m(m-1)=0

or m=0,m=1

which are all distinct and thus the corresponding 'c' is given by

STEP 3. $c = \frac{-\phi_2(m)}{\phi_3'(m)}$

$= \frac{-(m^2-2m-3)}{2m-1}$

Now for m=0, c=-3

$y = mx + c$ is given by

y=-3 or y+3=0 is an Asymptote.

For $m = 1, c = \frac{-(1-2-3)}{1} = 4$

Thus $y = mx + c$ or $y = x + 4$ is an Asymptote.

Hence, all the Asymptotes of the given curve are x+1=0, y+3=0 and y=x+4.

OBSERVATION

Further, the given equation can be written as

$$y^2(x+1) + x^2(-3-y) - 2xy$$

1. The Asymptote parallel to x-axis is given by equating to zero the coefficient of the

highest degree trem of x i.e. -3-y=0

or y+3=0.

2. The Asymptote parallel to y-axis is given by equating to zero the coefficient of the highest degree term of y i.e. x+1=0.

Example 9.6. Find the Asymptotes if any of the curve
$4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$

Solution. We have

$\phi_3(x, y) = 4x^3 - 3xy^2 - y^3$

$\phi_2(x, y) = 2x^2 - xy - y^2$

STEP 1. Putting x=1 , y=m in the above polynomials we get,

$\phi_3(m) = 4 - 3m^2 - m^3$

$\phi_2(m) = 2 - m - m^2$

STEP 2. The slopes of the Asymptotes are given by,

$\phi_3(m) = 0$

$m^3 + 3m^2 - 4 = 0$

$(m - 1)(m + 2)^2 = 0$

or m=1,-2,-2.

STEP 3. For m=1, corresponding value of ‘c’ is calculated by,

$c = \frac{-\phi_2(m)}{\phi_3'(m)}$

$c = \frac{-(-m^2-m+2)}{-3m^2-6m}$

or $c = \frac{m^2+m-2}{-3m(m+2)}$

For m=1, $c = \frac{2-2}{-3(3)} = 0$

thus, y=mx+c or y=x is an Asymptote.

STEP 4. Now for repeated roots m=-2,-2

‘c’ is calculated by

$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m)) + \phi_1(m) = 0$

or $\frac{c^2}{2}(-6 - 6m) + c(-2m - 1) = 0$

or $c^2(-3 - 3m) + c(-2m - 1) = 0$

or $3c^2(m + 1) + c(2m + 1) = 0$

puting m=-2 in the above quadratic equation in 'c' we get,

$3c^2(-1) + c(-3) = 0$

or $3c^2 + 3c = 0$

or 3c(c+1)=0

$\Rightarrow c = 0, c = -1$

Thus the required Asymptotes for m=-2, c=0 ,is y=-2x,
and for m=-2 ,c=-1 is y=-2x-1.

Hence all the three Asymptotes are

y=x, y=-2x, y=-2x-1.

Example 9.7. An Asymptotes touches the curve in atleast (i) one point (ii) two points (iii) three points (iv) one of these.

Solution. Since the Asymptote touches the curve at infinity hence the correct answer is (iv).

Example 9.8. Find the Asymptotes for the curve, if any?

$$x^4 + y^4 - 2x^2y^2 - 4x^2 + x = 0$$

Solution. We have
$\phi_4(x, y) = x^4 + y^4 - 2x^2y^2$
$\phi_3(x, y) = 0$
$\phi_2(x, y) = -4x^2$
$\phi_1(x, y) = x$

STEP 1. putting x=1,y=m in the above polynomials,we get
$\phi_4(m) = 1 + m^4 - 2m^2$
$\phi_3(m) = 0$
$\phi_2(m) = -4$

$\phi_1(m) = 1$

STEP 2.The slopes for the required Asymptotes are given by $\phi_4(m) = 0$
or $m^4 - 2m^2 + 1 = 0$
$\Rightarrow (m^2 - 1)^2 = 0$
$= (m^2 - 1)(m^2 - 1) = 0$
$= m = +1, -1, +1, -1$

STEP 3. For repeated root m=-1, 'c' is given by
$\frac{c^2}{2!}\phi_4''(m) + c\phi_3'(m) + \phi_2(m) = 0$
or $\frac{c^2}{2}(12m^2 - 4) + c(0) - 4 = 0$
or $c^2(6m^2 - 2) - 4 = 0$
or $c^2(3m^2 - 1) - 2 = 0.$
for m=-1, 'c' is ginen by
$c^2(3 - 1) - 2 = 0$
$\Rightarrow 2c^2 - 2 = 0$
$\Rightarrow c^2 - 1 = 0$
$\Rightarrow c = \pm 1$
y=-x+1 and y=-x-1.

STEP 4. For m=1, we have
$c^2(3 - 1) - 2 = 0$
or $2c^2 - 2 = 0$
or $c^2 = 1$
$\Rightarrow c = \pm 1$
y=x+1 and y=x-1.
Hence y=x+1, y=x-1, y=-x+1, y=-x-1 are the four Asymptotes of the given curve.

NOTE. In the above question in the equation $x^4 + y^4 - 2x^2y^2 - 4x^2 + x = 0$
There are no Asymptotes parallel to x-axis and y-axis. Since the coefficients of highest degree terms of x and y are constants.

Example 9.9. Find the Asymptotes of the curve given by the equation.

$$y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0$$

Solution. The equation of the curve is of degree 4. hence it cannot have more than four Asymptotes. From the equation of the curve, we have

$\phi_4(x, y) = y^4 - 2xy^3 + 2x^3y - x^4$
$\phi_3(x, y) = -3x^3 + 3x^2y + 3xy^2 - 3y^3$
$\phi_2(x, y) = -2x^2 + 2y^2$
$\phi_1(x, y) = 0$

STEP 1. Putting x=1, y=m in the above polynomials. We get
$\phi_4(m) = m^4 - 2m^3 + 2m - 1$

$= (m-1)(m^3 - m^2 - m + 1)$
$= (m-1)^2(m^2 - 1)$
$= (m-1)^3(m+1)$
$\phi_3(m) = -3m^3 + 3m^2 + 3m - 3$
$\phi_2(m) = 2(m^2 - 1)$
$\phi_1(m) = 0$

STEP 2. For the slopes of the Asymptotes y=mx+c, we have the roots of the equation $\phi_4(m) = 0$
$\Rightarrow (m-1)^3(m+1) = 0$
$\Rightarrow m = 1, 1, 1 and -1$
Thus, of the four Asymptotes, three have equal slopes or are parallel.

STEP 3. For m=-1, 'c' is given by
$c = \frac{-\phi_3(m)}{\phi_4'(m)}$
$= \frac{-3(-m^3+m^2+m-1)}{3(m-1)^2(m+1)+(m-1)^3 \cdot 1}$
$= \frac{-3(-(-1)^3)+(-1)^2-1-1}{0+(-2)^3}$
$= 0.$
Therefore for m=-1, c=0 the Asymptote y=mx+c is given by y=-x.

STEP 4. Now for repeated roots m= 1,1,1 the corresponding value of 'c' is calculated by $\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$
or $\frac{c^3}{6}(24m - 12) + \frac{c^2}{2}(-18m + 6) + c \cdot 4m = 0$
or $c[(4m-2)c^2 + (-9m+3)c + 4m] = 0$
putting m=1, c=0 and $2c^2 - 6c + 4 = 0$
or (c-1)(c-2)=0
thus the three values of c for m=1 are c=0,1,2.
Therefore,the four Asymptotes are
y=-x (m=-1,c=0)
y=x (m=1,c=0)
y=x+1 (m=1, c=1)
y=x+2 (m=1, c=2)

In-text Exercise 9.1. Find the asymptotes of the following curves :

1) $x^3 + 3xy^2 + y^2 + 2x + y = 0$

2) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

3) $y^2(a^2 - x^2) = x^4$

4) $x^2y^3 + x^3y^2 = x^3 + y^3$

5) $xy^3 +^3 y = a^4$

6) $(x^3 + a^3)y = bx^3$

7) Prove that the four asymptotes of the curve $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$ form a square.

9.4 Concavity of a Curve

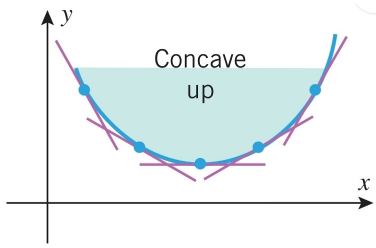


Figure 9.2: Concave Up

Definition 9.3. (Concave upwards at a point) A portion of the curve on both sides of a point lies above any tangent line drawn to it on the point, then the curve is said concave upwards at P.

Definition 9.4. Concave downwards at a point) A portion of the curve on both sides of a point lies below any tangent line drawn to it on the point, then the curve is said concave downwards at P.

Definition 9.5. (Concave upwards on an Interval) A curve $y = f(x)$ is said to be concave upwards in an interval, if it is concave upwards at every point of that interval. That is, if the curve bends upwards on that interval. In otherwords, the portion of the curve corresponding to the interval, lies above the tangent line at any point of the curve corresponding to the interval as shown in figure 9.2.

Definition 9.6. (Concave downwards on an Interval) A curve $y = f(x)$ is said to be concave downwards in an interval, if it is concave downwards at every point of that interval. That is, if the curve bends downwards on that interval. In otherwords, the portion of the curve corresponding to the interval, lies below the tangent line at any point of the curve corresponding to the interval as shown in figure 9.3.

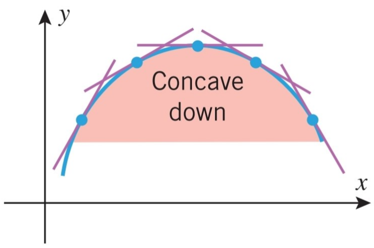


Figure 9.3: Concave Down

9.4.1 Criteria for Concavity

Let $y = f(x)$ be a function defined on an open interval I, such that $f''(x)$ exists for $x \in I$. We know that the sign of the first order derivative $f'(x)$ indicates whether the curve $y = f(x)$ is increasing or decreasing on I. For checking the concavity (or convexity) of $y = f(x)$ on I using the second order derivative $f''(x)$ for $x \in I$. We have the following criteria:

(i) The curve $y = f(x)$ is concave upwards on I if $f''(x) > 0 \, \forall x \in I$

(i) The curve $y = f(x)$ is concave downwards on I if $f''(x) < 0 \, \forall x \in I$

Example 9.10. For $f(x) = x^3 - 3x^2 + 1$, find the intervals on which $f(x)$ is

(i) concave upwards

(i) concave upwards

Solution. We have,

$$f(x) = \quad x^3 - 3x^2 + 1 \Rightarrow f'(x) = \quad 3x^2 - 6x \Rightarrow f''(x) = \quad 6(x-1)$$

For $x < 1$, $f''(x) < 0$.
Therefore, $f(x)$ is concave downwards in $(-\infty, 1)$.
For $x > 1$, $f''(x) > 0$.
Therefore, $f(x)$ is concave inwards in $(1, \infty)$.

Example 9.11. let $f(x) = x^3$, then find the interval where it is concave up and concave down.

Solution. $f(x) = x^3$

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$
$$f''(x) = 0$$
$$\implies 6x = 0$$
$$\implies x = 0$$

Case I : $x \in (-\infty, 0)$

$$f''(x) = 6x < 0$$

$\implies$ f is concave down on the interval $(-\infty, 0)$

Case II : $x \in (0, \infty)$

$$f''(x) = 6x > 0$$

$\implies$ f is concave up on the interval $(0, \infty)$

9.5 Point of Inflexion

Definition 9.7. Point of Inflexion of a curve A point on the curve $y = f(x)$ at which the curve changes its concavity from upwards to downwards or from downwards to upwards is called as a point of Inflexion.

For example, $f(x) = x^{1/3}$ has $(0, 0)$ as an inflexion point which is depicted in Fig. 10.4.

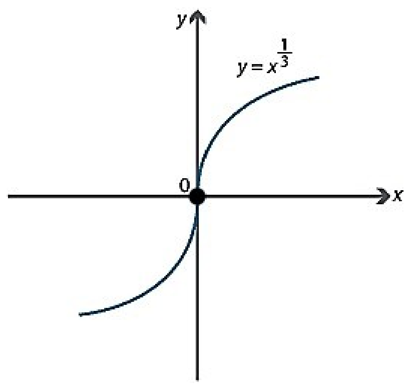


Figure 9.4: Inflexion Point

Note.
- A curve changes its shape at a point of inflexion.
- A curve crosses the tangent line at a point of inflexion.

9.5.1 Criteria for Point of Inflexion

If $P(a, f(a))$ is a point of inflexion of the curve $y = f(x)$, then $f''(a) = 0$ or $f''(a)$ does not exist.

Example 9.12. Discuss the concavity of $f(x) = x^3$ and find its point of inflexion.

Solution. We have $f(x) = x^3$, $x \in \mathbb{R}$.
then $f'(x) = 3x^2$ and $f''(x) = 6x$. Now, $f''(x) = 6x$ imply that, $f''(x) < 0$ for $x \in (-\infty, 0)$ and $f''(x) > 0$ for $x \in (0, \infty)$. Therefore, the curve $f(x) = x^3$ is concave downwards on $(-\infty, 0)$ and concave upwards on $(0, \infty)$ and it has a point of inflexion at $x = 0$, as $f''(x)$ changes its sign as x passes through $x - 0$.

From the above mentioned criteria, we note that if $f''(a) = 0$, then the curve $y = f(x)$ may not have point of inflexion at $x = a$.
For example, Consider the function $f(x) = x^4$. Then

$$f''(x) = 12x^2 \Rightarrow f''(0) = 0$$

But $x = 0$ is not a point of inflexion for $f(x) = x$.
Similarly, if $f''(x)$ does not exist at a point $x = a$, the the curve $y = f(x)$ may have a point of inflexion at $x = a$. For example, the function $f(x) = x^{1/3}$ has a point of inflexion at $x = 0$, but $f''(0)$ does not exist.

Example 9.13. Find the intervals on which $f(x) = x^3 - 3x^2 + 1$ is (i) Concave up (ii) Concave down (iii) Locate all the points of inflexion.

Solution. As $f(x) = x^3 - 3x^2 + 1$, then $f'(x) = 3x^2 - 6x = 3x(x-2)$ and $f''(x) = 6x - 6 = 6(x-1)$
For $x < 1$, $f''(x) = 6(x-1)$ is negative thus f is concave downwards in $(-\infty, 1)$, and
For $x > 1$, $f''(x) = 6(x-1)$ is positive thus f is concave upwards in $(1, \infty)$. We can evealuate points of inflexion by $f''(x) = 0$

$$\Rightarrow 6(x-1) = 0 \Rightarrow x = 1$$

Clearly, $x = 1$ is the inflexion point since $f(x)$ changes from concave down to concave up at x=1.

9.6 Summary

Following points have been discussed in this lesson

- An asymptote of a curve $y = f(x)$ is a straight line if the perpendicular distance between the line and the point $A(x, y)$ on the curve approach to 0 as x or y or both approach to infinity.
- There are three kinds of asymptotes of a curve $y = f(x)$, namely,
  - Horizontal Asymptote
  - Vertical Asymptote
  - Oblique Asymptote
- Concavity of $y = f(x)$ at a point (concave upwards and concave downwards)
- Concavity of $y = f(x)$ in an interval (concave upwards and concave downwards)
- Criteria for checking the concavity of a curve $y = f(x)$.
- Point of inflection and criteria for obtaining point of inflection for given curve.

9.7 Self Assessment Exercise

1. Find the asymptotes for the following curves:
   (a) $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$
   (b) $(x+y)^2(x+2y+2) = x + 9y - 2$
   (c) $\frac{a^2}{x^2}+\frac{b^2}{y^2} = 1$
   (d) $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$
   (e) $x^3 + 3xy^2 - 4y^3 - x + y + 3 = 0$

(f) $x^3 - 2x^2y + xy^2 - x^3 - xy + 2 = 0$

(g) $x^2(x-y)^2 + a^2(x^2 - y^2) = a^2xy$

(h) $y^2(x-2a) = x^3 - a^3$

(i) $x^2y + xy^2 + xy + y^2 + 3x = 0$

(j) $(y-a)^2(x^2 - a^2) = x^4 + a^4$

(k) $x^3 + y^3 = 3axy.$

2. Discuss the concavity of $f(x) = e^x$ and $log(x)$.

3. Find the intervals in which $f(x) = x^5$ is concave up,concave down. Also find the points of inflexion.

9.8 References

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Lesson - 10

Curve Tracing

Structure

10.1 Learning Objectives

The importance of this chapter lies in the fact that it enables the students to learn the following:

- How to draw the various algebraic curves.
- Multiple points of the curve such as *Node*, *Cusp*, *Isolated points*.
- Symmetry of the curve.
- Asymptotes to a curve by inspection.
- Tangents at the origin to a curve.
- Region of existence of curve.
- Region of absence of curve.
- Point of intersection with coordinate axes.

After studying this chapter students will be able to draw rough sketch of various algebraic curves and will learn about their properties.which helps them in solving many mathematical problems.

10.2 Introduction

Curve tracing is a method of drawing a rough sketch or shape of a curve by the study of some of its important characteristics such as symmetry, origin, point of intersection with coordinate axes, asymptotes, tangents, multiple points, region of absence of curves, region of existence of curves. Knowledge of curve tracing is useful in application of integration for computing areas, lengths, volume of solids of revolution and surface of solids of revolution in other chapters. It is very important and helpful to know the shape of a curve represented by the given equation. Curve tracing helps us to draw the rough sketch of the curve which further helps us in solving various mathematical problems. In this chapter, we learn to draw various algebraic curves using some standard rules and their properties.

10.3 Tangent and Normal to a Curve

If θ be the angle which the tangent at any point P(a,b) on the curve y=f(x) makes with x-axis, then $tan\theta =\frac{dy}{dx} = f'(x)$ and is called as the slpoe 'm' of the tangent line to the curve at the point P(a,b).It now follows that

Definition 10.1. The equation of a *tangent* to a given curve at point P(a,b) is given by:-

$$y - b = \frac{dy}{dx}(x - a)$$

where, $\frac{dy}{dx}$ is the slope of the tangent.

Definition 10.2. The *normal* to the curve y=f(x) at any poit P(a,b) is the straight line passing through the point P(a,b) and is perpendicular to the tangent to the curve at that point. Slope of the Normal is given by $-\frac{dx}{dy}$.

Where slope of the tangent was $\frac{dy}{dx}$.

Hence, the equation of the Normal to the curve y=f(x) at P(a,b) is given by

$$y - b = -\frac{dx}{dy}(x - a)$$

Example 10.1. Find the equation for the tangent and normal to the parabola $y = x^2$ at the point P(1,1).

Solution. Equation of the parabola is $y = x^2$.
which gives $\frac{dy}{dx} = 2x$
now at the point (1,1) we have

$\frac{dy}{dx} = 2$
Thus the equation of the tangent to the given curve at (1,1) is given by
$y - 1 = 2(x - 2)$

or $y - 1 = 2x - 2$

or $y = 2x - 1$

Equation of normal at (1,1)
$y - 1 = -\frac{1}{2}(x - 1)$

or $2y - 2 = -x + 1$

or $x + 2y = 3$

Example 10.2. Find the equation of tangent and normal for the curve $y^2 = 4x + 5$.

Solution. Given curve $y^2 = 4x + 5$

Differentiating w.r.t. 'x', we get

$2y\frac{dy}{dx} = 4$

$\Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$

$\Rightarrow \frac{dy}{dx} = 2$ at (-1,1)

Now, equation of the tangent line at (-1,1) having slope 2 is given as

$y - 1 = 2(x - 1)$

or $y - 1 = 2x + 2$

or $y = 2x + 3$

Also, equation of normal at (-1,1) is given by

$y - 1 = -\frac{dx}{dy}(x + 1)$

$\Rightarrow (y - 1) = -\frac{1}{2}(x + 1)$

or $2y - 2 = -x - 1$

or $x + 2y = 1$

10.4 Tracing of Cartesian Curves

I **SYMMETRY**

(a) A curve is symmetrical about x-axis if all the powers of y in the equation of the given curve are even. or if $f(x, -y) = f(x, y)$
Example $y^2 = 4ax$

(b) A curve is symmerical about y-axis if all the powers of x in the equation of the given curve are even. or if $f(-x, y) = f(x, y)$
Example $x^2 = 4ay$

(c) A curve is symmetrical about the line $y = x$ if the equation of the given curve remains unchanged on interchanging x and y. i.e. f(x,y)=f(y,x)
Example $x^3 + y^3 = 3axy$

II **THE ORIGIN**

If the equation of the curve is satisfied on putting x=0 and y=0 then we say that origin lies on the curve or in other words if there is no constant term in the equation , we say that curve passes through origin.

Short-cut Rule for Finding Tangent at the Origin

If the curve passes through the origin we can find the equation of the tangent to the curve at the origin by equating to zero the lowest degree term in the equation.

Example 10.3. Find the tangent at the origin to the curve $y^2 = 4ax$

Solution. In the above equation,

the tangent is given by putting lowest degree term equal to zero i.e.$4ax = 0$ thus $x = 0$ is the tangent to the curve at the origin.

Example 10.4. Find the tangent at the origin to the curve $y = x^3$

Solution. The curve $y = x^3$ has $y = 0$ as tangent to the curve at the origin.since the lowest degree term in the equation is y.

Example 10.5. Find the tangent at the origin to the curve

$$a^2(x^2 - y^2) = x^2y^2$$

Solution. Lowest degree term in the above equation is $a^2(x^2 - y^2)$. Tangent at the origin is given by $(x^2 - y^2) = 0 \Rightarrow y = \pm x$

Thus $y = x$ and $y = -x$ are the two tangents at the origin.

III **MULTIPLE POINTS**

A point on a curve through which two branches of a curve passes is called as *double point* or *multiple point*.

clearly a curve has two tangents at a double point , one for each branch.

Double point is further classified as *Node* , *a Cusp* , *isolated or conjugate point* according as the two tangents are Real and distinct , Coincident or Imaginary respectively.

Nature of Double Point at the Origin

If there are two or more tangents to the curve at the origin , it is called a multiple point.

Origin (Double Point) is classified as:

(a) Node: two tangents at the origin are real and distinct.

(b) Cusp: two tangents at the origin are real and coincident.

(c) Isolated : two tangents at the origin are imaginary.

Example 10.6. Find the nature of double point at the origin of the curve $y^2(a^2 + x^2) + x^2(a^2 - x^2) = 0$.

Solution. $x = 0, y = 0$ satisfies the given equation thus curve passes through origin.

The lowest degree term in the equation is $y^2a^2 + x^2a^2$

thus tangents at the origin is given by

$y^2a^2 + x^2a^2 = 0$
$\Rightarrow y = \pm ix$(imaginery)

Hence, origin is a isolated point.

Example 10.7. Find the nature of double point at the origin of the curve $y^2(a-x) = x^3$

Solution. $x = 0, y = 0$ satisfies the given equation thus curve passes through origin.

now lowest degree term in the equation is y^2a thus tangent at the origin is given by $y^2a = 0$ or $y = 0$(real and coincident).

Hence, origin is a cusp.

Example 10.8. find the nature of double point at the origin of the curve $y^2(x + a) = x^2(3a - x)$

Solution. Clearly, $x = 0, y = 0$ satifies given equation thus curve passes through origin.

lowest degree term in the equation is given by $ay^2 = 3ax^2$

thus tangent at the origin is given by $ay^2 - 3ax^2 = 0$

or $y = \pm x\sqrt{3}$(real and distinct)

Therefore, origin is a Node.

IV POINT OF INTERSECTION WITH COORDINATE AXES

(a) Find the points where the curve intersects the x-axis and the y-axis seperately. take x=0 for the intersection of the given curve with y-axis and take y=0 for the intersection of the curve with x-axis. In this way, we get points on the axes through which a given curve may pass.

(b) Find the tangent to the curve at its point of intertsection with the coordinate axes by first shifting the origin to this point and then equating to zero the lowest degree term.

V **REGION OF ABSENCE OF CURVE**

If possible find the region of the plane where no part of the curve lies. such a region is obtained on solving the given equation for y^2 in terms of x (or x^2 in terms of y). Suppose $y^2 < 0$ for $x > a$ .Similarly if $x^2 < 0$ for $y > b$, then the curve does not lies in the region $y > b$.

VI **ASYMPTOTES**

Find out the Asymptotes of the curve,if any by inspection.

Recall that Asymptotes parallel to x-axis is given by equating to zero coefficient of highest power of x and Asymptotes parallel to y-axis is given by equating to zero coefficient of highest power of y.

VII **POINT OF INFLECTION AND CONCAVITY**

Check the points of inflexion and concavity of the curve by finding

(a) $f''(x) = 0$ (For points of inflexion)

(b) $f''(x) > 0$ (Curve concave upwards)

(c) $f''(x) < 0$ (Curve concave downwards)

10.5 Examples on Curve Tracing

Example 10.9. Trace the curve $y^2(a^2 - x^2) = a^3x$

Solution. We need to follow the steps mentioned below.

Step 1. Symmetry: the given equation is even in y so curve is symmetrical about x-axis.

Step 2. Origin: Curve satisfies origin. clearly origin is a cusp.

Step 3. Tangent at the origin is given by $a^3x = 0$ or $x = 0$ i.e. y-axis.

Step 4. Asymptotes: Parallel to x-axis is given by $y^2a^2 = 0$ or $y = 0$.
Parallel to y-axis is given by $(a^2 - x^2) = 0$ or $x = \pm a$.

Step 5. For point of intersection with x-axis by put $y = 0$ in the equation. For point of intersection with y-axis put $x = 0$ in the given equation.clearly $(0, 0)$ is the only point of intersection.

Step 6. Region of absence of curve:

We can write $y^2 = \frac{a^3x}{a^2-x^2}$

for $0 < x < a$, y^2 is +ve

for $-a < x < 0$, y^2 is -ve

for $x > a$, y^2 is -ve

for $x < -a$, y^2 is +ve

Step 7. To trace it more accurately , take some points such as $x = \frac{a}{2}, \frac{a}{3}, \frac{-a}{2}, \frac{-a}{3}$,...... find y corresponding to it and then trace.

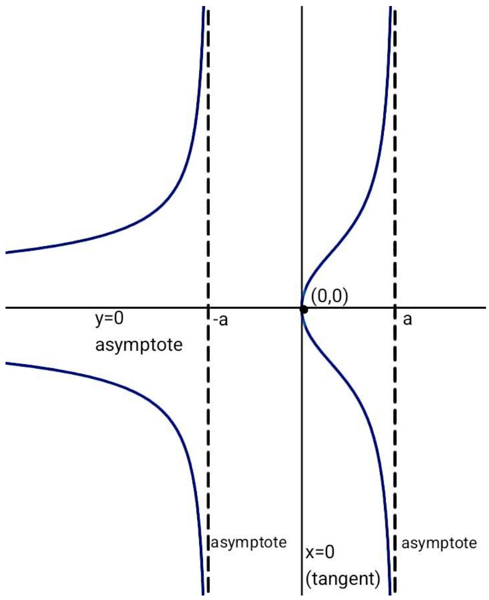

Figure 10.1: Curve of equation $y^2(a^2 - x^2) = a^3x$

Example 10.10. Trace the curve $y(x^2 + a^2) = a^3$

Solution. We need to follow the steps mentioned below.

Step 1. Symmetry: the given equation is even in x,so curve is symmetrical about y-axis.

Step 2. Origin: Put $x = 0$,$y = 0$ in the given equation. It doesn't satisfies equation.Thus curve doesn't passes through origin.

Step 3. Asymptotes:

Parallel to x-axis is given by $ya^2 = 0$ or $y = 0$.

Parallel to y-axis is given by $(x^2 + a^2) = 0$ or $x = \pm ia$ (doesn't exist).

Step 4. Intersection with coordinate axes:

For intersection with x-axis put $y = 0$ in the equation, we get $a^3 = 0$ thus the curve doesn't intersect x-axis.

For intersection with y-axis put $x = 0$ in the given equation, we get $ya^2 - a^3 = 0$

$\Rightarrow a^2(y - a) = 0 \Rightarrow y = a$, so $(0, a)$is the point of intersection with y-axis.

Step 5. For tangent at $(0, a)$,put $y = y + a$ in the given equation.

We get $(y + a)(x^2 + a^2) = a^3 \Rightarrow yx^2 + ya^2 + ax^2 + a^3 = a^3$.

Lowest degree term is given by $ya^2 = 0 \Rightarrow y = 0$(x-axis) is the tangent at $(0, a)$.

Step 6. Region: $x^2 = \frac{a^3 - ya^2}{y}$ or $x^2 = \frac{a^2(a-y)}{y}$

(a) For $y > a$, x^2 is -ve (no portion of curve lies)

(b) For $0 < y < a$, x^2 is +ve (curve lies)

(c) For $y < 0$ say $y = -2a$, x^2 is -ve (no portion of curve lies)

Step 7. Further, give values to $x = \frac{a}{2}, a, 2a$,.... to trace the curve more precisely.

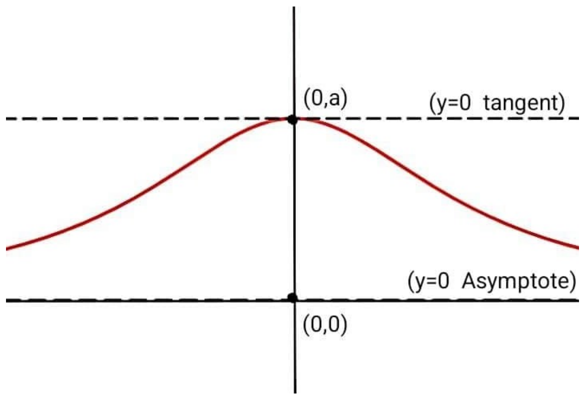

Figure 10.2: Curve of equation $y(x^2 + a^2) = a^3$

Example 10.11. Trace the curve

$$y^2(a+x) = x^2(3a-x)$$

Solution. We need to follow the steps mentioned below.

Step 1. Symmetry: the given equation is even in y,so curve is symmetrical about x-axis.

Step 2. The given equation satisfies $x = 0$ and $y = 0$ so curve passes through origin.

Step 3. Further, tangent at the origin is given by putting lowest degree term equal to zero i.e. $a(y^2 - 3x^2) = 0$

$\Rightarrow y^2 = 3x^2$
$\Rightarrow y = \pm\sqrt{3}x$

Thus, two real and distinct tangents exist at origin so origin is a Node.

Step 4. Asymptotes:

Parallel to x-axis : doesn't exists.

Parallel to y-axis : $x + a = 0 \Rightarrow x = -a$.

Step 5. For point of intersection with x-axis put $x = 0$ we get $y = 0$.

For point of intersection with y-axis put $y = 0$ we get $x = 0$,$x = 3a$.

So, (0,0) and (3a,0) are the points where the curve touches the coordinate axes.

Step 6. Tangent at (3a,0)

Let us change the origin from (0,0) to (3a,0)

put $x = x + 3a$ in the equation and then check for tangent at (3a,0).

$y^2(x + 4a) = (x + 3a)^2(-x)$

or, $xy^2 + 4ay^2 = (x^2 + 9a^2 + 6ax)(-x)$

or, $xy^2 + 4ay^2 = -x^3 - 9a^2x - 6ax^2$

Now put lowest degree term equal to zero.

i.e. $9ax^2 = 0 \Rightarrow x = 0$ (y-axis).

Thus y-axis is the tangent at point (3a,0).

Step 7. Region: We have $y^2 = \frac{x^2(3a-x)}{x+a}$

(a) $y^2 < 0$ when $x < -a$ i.e. for $x = -2a, -3a$, ...

(b) $y^2 < 0$ when $x > 3a$

Thus no portion of the curve lies in a) and b).

Step 8. Find some more points on the curve by giving values to x such as $x = \frac{a}{2}, 2a$,.... to trace the curve more precisely.

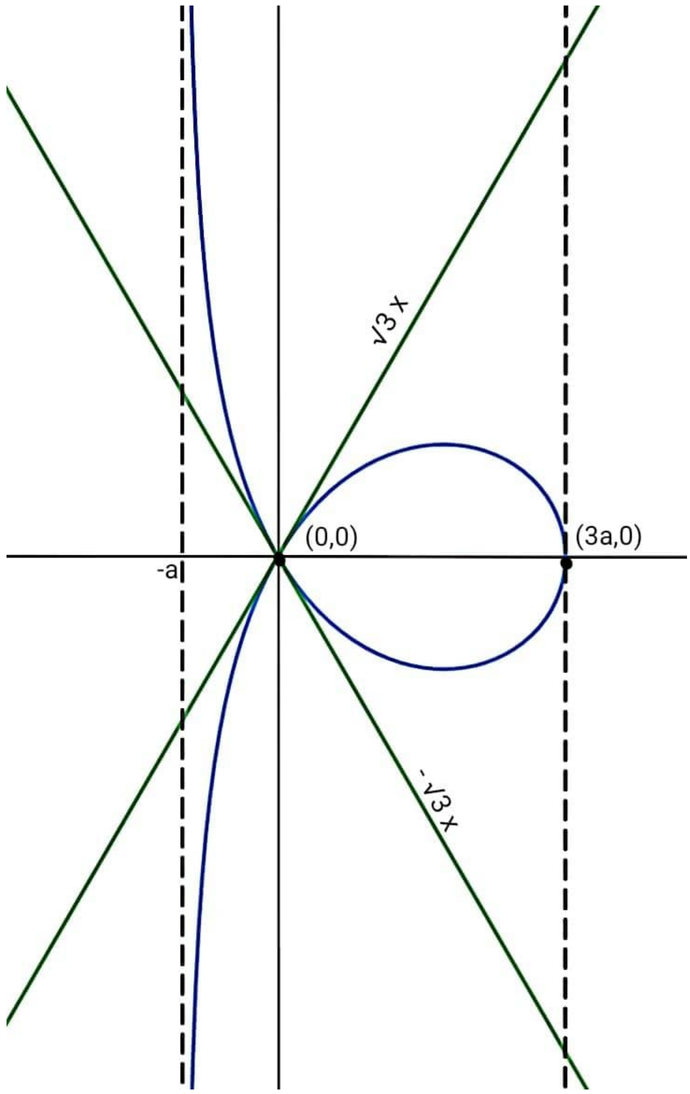

Figure 10.3: Curve of equation $y^2(a+x) = x^2(3a-x)$

Example 10.12. Trace the curve $x^2(x^2 + y^2) = a^2(x^2 - y^2)$.

Solution. We need to follow the steps mentioned below.

Step 1. Curve is symmetrical about about both the axes. Since powers of x and y are both even in the equation.

Step 2. $x = 0$, $y = 0$ satisfies the given equation thus curve passes through origin.

Step 3. Tangents at the origin is given by putting lowest degree term equal to zero,

i.e. $x^2 - y^2 = 0$.

$\Rightarrow y = \pm x$. We get two real and distinct tangent at the origin. Thus, origin is a Node.

Step 4. Asymptotes:

Parallel to x-axis doesn't exist.

Parallel to y-axis is given by $x^2 + y^2 = 0 \Rightarrow x = \pm ia$ (does not exist).

Step 5. Point of intersection:

With x-axis put y=0 which gives $x = 0$, $x = \pm a$

With y-axis put x=0 which gives y=0.

Thus, (0,0),(a,0),(-a,0) are the point of intersection with coordinate axes.

Step 6. Tangent at (a,0) and (-a,0)

Let us shift the origin to the point (a,0) by putting x = x+a, in the given equation. Also, shift the origin to the point (-a,0) by putting x = x-a, then find the tangents.

We see that y-axis is the only tangent.

Step 7. Region:

We have $y^2 = \frac{x^2(a^2-x^2)}{x^2+a^2}$

$y^2 < 0$ when $x^2 > a^2$ or $x > a$ and $x < -a$.

Thus, no portion of the curve lies where $x > a$ and $x < -a$.

Step 8. We can further trace the curve by taking more points like $x = \frac{a}{2}, \frac{a}{3}, \frac{-a}{2}, \frac{-a}{3}, ...$

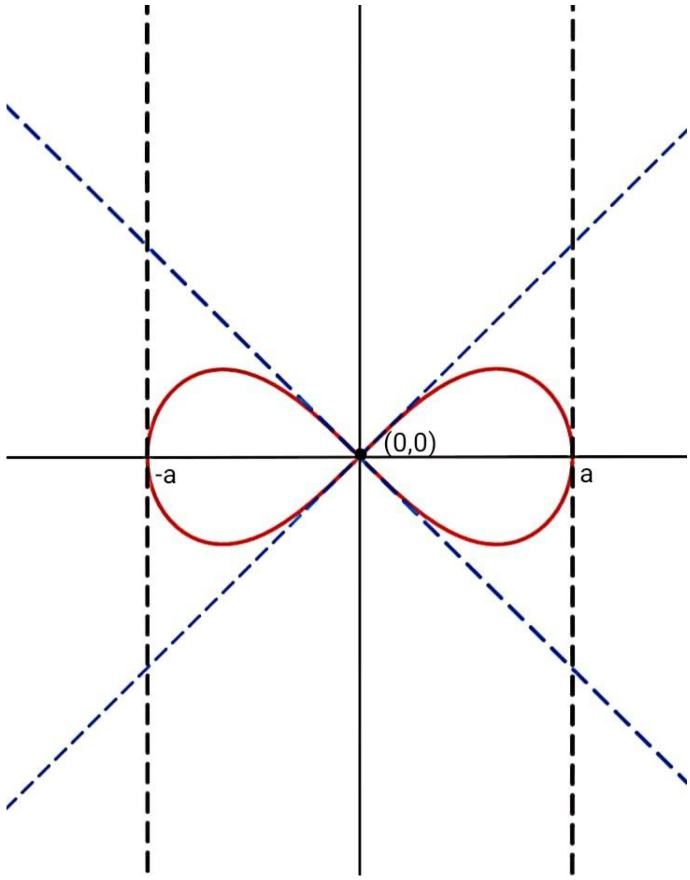

Figure 10.4: Curve of equation $x^2(x^2 + y^2) = a^2(x^2 - y^2)$

Example 10.13. Trace the curve
$ay^2 = x(a^2 - x^2), \qquad a > 0$

Solution. We need to follow the steps mentioned below.

Step 1. Symmetry: the given equation of curve is even in y, so curve is symmetrical about x-axis.

Step 2. $x = 0$, $y = 0$ satisfies the equation thus the curve passes through the origin.

Step 3. Tangent at the origin can be found by putting lowest degree term equal to zero in the equation, i.e.

$xa^2 = 0$
$\Rightarrow x = 0$ (y-axis)

Thus, y-asis is the tangent at the origin.

Step 4. Asymptote:

Parallel to x-axis does not exists.

Parallel to y-axis does not exists.

Step 5. Point of intersection:

For intersection with x-axis put $y = 0$ in the equation, we get

$x(a^2 - x^2) = 0 \Rightarrow x = 0, \pm a$

Step 6. Tangent at $(a, 0)$ and $(-a, 0)$.

Shift the origin from $(0, 0)$ to $(a, 0)$ by substituting $x = x + a$ in the given equation, we get

$ay^2 = (x + a)[a^2 - (x + a)^2]$

$\quad = (x + a)(a^2 - x^2 - a^2 - 2ax)$

$\Rightarrow ay^2 = (x^3 + 2ax^2 + ax^2 + 2a^2x)$

Tangent at $(a, 0)$ is given by putting lowest degree term equal to zero in above equation,

i.e. $2a^2x = 0 \Rightarrow x = 0$.

Thus, x=0 (y-axis) is the tangent at $(a, 0)$

Similarly, at $(-a, 0)$ y-axis is the tangent.

Step 7. Region:

We can write the given equation as $y^2 = \frac{x(a^2-x^2)}{a}$

(a) For $0 < x < a$, y^2 is +ve (curve lies)

(b) For $-a < x < 0$, y^2 is -ve (curve does not lie)

(c) For $x > a$, y^2 is -ve (curve does not lie)

(d) For $x < -a$, y^2 is +ve (curve lies)

Step 8. Further to trace the curve more accurately give values to x like $x = \frac{a}{2}, -2a, -3a$,......and find the corresponding values of y. Plot these points.

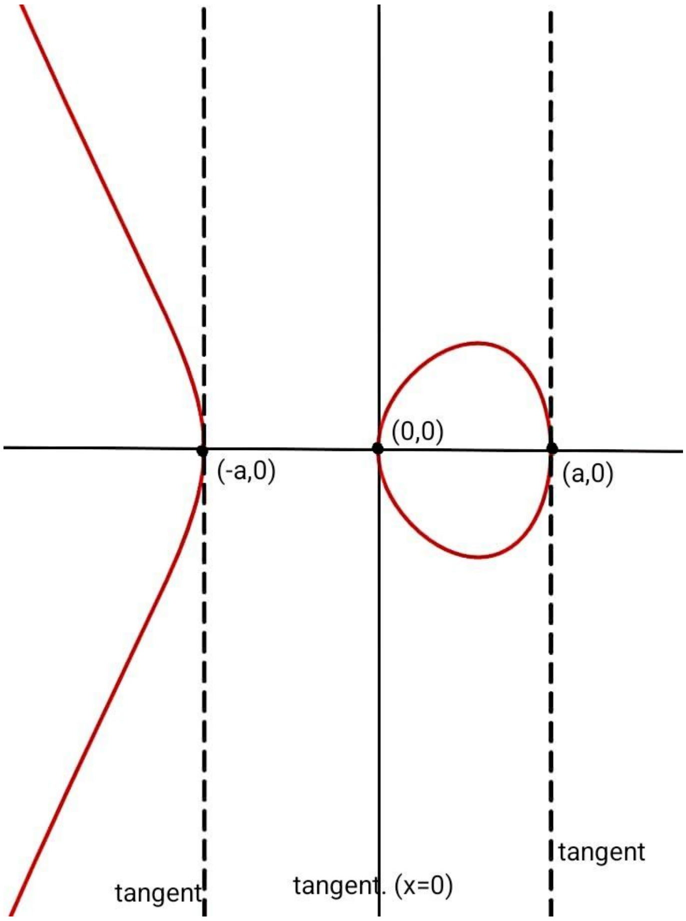

Figure 10.5: Curve of equation $ay^2 = x(a^2 - x^2), \qquad a > 0$

10.6 Exercise

Trace the following curves:

1. $a^2x^2 = y^3(2a - y)$
2. $4ay^2 = x(x - 2a)^2$
3. $3ay^2 = x(x - a)^2$
4. $xy^2 + (x + a)^2(x + 2a) = 0$
5. $(x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = a^{\frac{2}{3}}$
6. $y(x^2 + 4a^2) = 8a^3$
7. $y^2x = a^29a - x)$
8. $y^2(2a - x) = x^3$
9. $y^2(a^2 + x^2) = x^4$
10. $y^2x^2 = x^2 + a^2$

10.7 Summary

Curve tracing is a technique of drawing rough sketches of the algebraic curves by following some standard steps like symmetry of the curve about axes, origin of the curve, shifting origin to the point of intersection of the curves with x-axis and y-axis.

Finding tangents at the origin and at the point of intersection of the curve with the coordinate axes. Finding multiple or double points of a curve such as Node, cusp, isolated point. Finding asymptotes to a curve, region of absence and existence of a curve.

By following these very basic steps students can trace various algebraic curves and learn their properties.which enhances their knowledge of drawing curves and solving various problems of mathematics.

10.8 References

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