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Semester-I

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THEORY OF EQUATIONS AND SYMMETRIES

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Contents

Lesson - 1

General Properties of Polynomial Equations

Structure

1.1 Learning Objectives

Student will be able to

- divide a polynomial by a linear polynomial without performing actual division.
- find the number of positive, negative and imaginary roots of a polynomial without actually solving it.
- understand the basic properties of roots of a polynomial equation.

1.2 Introduction

One of the oldest problems in mathematics is solving algebraic equations, in particular finding the roots of the polynomial equations. The ancient mathematicians solved some particular problems and there was no generality. In this chapter we shall study some important theorems related to polynomial equations such as the Remainder theorem, Factor theorem, Fundamental theorem of algebra, etc. "The Fundamental theorem of algebra" which state that every polynomial of degree ≥ 1 has at least one zero was first proved by the famous German mathematician Karl Fredrich Gauss. Also we will learn about Descartes' rule of signs for counting the positive and negative roots of a polynomial equation without actually finding the roots. Using these ideas we will reach our goal of solving polynomial equations of certain type. Here we will also understand some properties of the roots of a polynomial equation. But before discussing about these theorems, we start a brief discussion of the polynomial equations and degree of the polynomial.

1.3 Polynomials

1.3.1 Polynomials

Polynomials is an algebraic expression that consist of variables (also called indeterminate) and coefficient. It involves only the operation of addition, subtraction, multiplication and non negative integer exponents(power) of variables.

Standard form of a polynomial is $a_0x^n + a_1x^{n-1} + a_2x^{n-2}.......a_n$, where $a_0, a_1, ...a_n$ are real constants called coefficients of the polynomial and x is a variable.

Example 1.1. 1. $4x^{-3} + 5$ is not a polynomial because x^{-3} have negative exponent (i.e. -3).

2. $5x^2 + 2y - 7$ is a polynomial in two variables(x and y) and $5, 2$ are the coefficient of x^2 and y respectively while -7 is constant.

3. $8x^2 - 5x + 6$ is a polynomial in one variables x and $8, -5$ are the coefficient of x^2 and x respectively while 6 is constant.

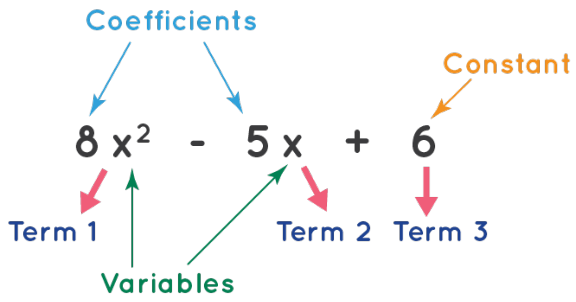

Figure 1.1: Polynomial with three terms.

1.3.2 Degree of a Polynomial

Degree of a polynomial is simply the highest exponent occurs in the polynomial equation. The degree of a term is the sum of the exponents of the variables that appears in it.

Example 1.2. 1. The polynomial $7x^3y^3+4x^2-9x+y$ has four terms. The first term has degree $6(3+3)$, the second term has degree 2 , third term has degree 1 and fourth term has degree 1. Therefore the polynomial has degree 6, which is the highest degree of any term.

2. $x^3 - 5x^2 + 6x + 2$ has degree 3.

3. $x^2y + 5x^2 + 3$ has degree 3

Definition 1.1. An equation of the form

$$p(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2}.......a_n = 0 \tag{1.1}$$

is called a polynomial equation of degree n.

Remark. 1. The polynomial equation of degree one, $a_0x + a_1 = 0$ is called linear equation.

2. The polynomial equation of degree two, $a_0x^2 + a_1x + a_2 = 0$ is called quadratic equation.

3. The polynomial equation of degree three, $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ is called cubic equation.

4. The polynomial equation of degree four, $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ is called bi quadratic equation.

1.4 General Properties of Polynomials and Equations

1.4.1 The Remainder Theorem

The remainder theorem or the polynomial remainder theorem is used to find the remainder when a polynomial is divided by a linear polynomial without actually carrying out the steps of the long divisions. It states that

Theorem 1.1 (The Remainder Theorem)**.** *When a polynomial $p(x)$ (whose degree is greater than or equal to one) is divided by a linear polynomial $(x-c)$ until a remainder independent of x is obtained then the remainder is given by $p(c)$ (which is a value of $p(x)$ when $x = c$.)*

Proof. Here dividend is $p(x)$ and divisor is $x - c$. Let quotient is $q(x)$ and remainder is denoted by r. As we know

$$\text{divident} = \text{divisor} \times \text{quotient} \ + \text{remainder}. \tag{1.2}$$

Then by using the dividend formula (1.2), we have

$$p(x) = (x - c)\ q(x) + r \tag{1.3}$$

By taking $x = c$ in equation (1.3), we get $p(c) = r$. □

Remark.
1. This theorem is also known as little Bézout's theorem
2. This theorem works only when the divisor is linear (this can be considered as one of its limitation)

Example 1.3. Without actual division , find the remainder when $x^4 - 3x^2 - x - 6$ is divided by $x + 3$.

Solution. Given a polynomial

$$p(x) = x^4 - 3x^2 - x - 6. \tag{1.4}$$

To find the remainder here, we need not to carry out the long steps of division. Here the divisor is $x + 3$ and dividend is $x^4 - 3x^2 - x - 6$.
Let

$$x + 3 = 0 \quad \text{then} \quad x = -3 \tag{1.5}$$

By putting $x = -3$ from equation (1.5) in the given polynomial (1.4), we get

$$\begin{aligned} p(-3) =& (-3)^4 - 3(-3)^2 - (-3) - 6 \\ =& 81 - 27 + 3 - 6 \\ =& 51. \end{aligned}$$

Verification:

$$
\begin{array}{r|llll}
 & x^3 - 3x^2 & +6x & -19 & \\
\hline
x+3 & x^4 & -3x^2 & -x & -6 \\
 & x^4 + 3x^3 & & & \\
\hline
 & -3x^3 - 3x^2 & & -x & -6 \\
 & -3x^3 - 9x^2 & & & \\
\hline
 & & 6x^2 & -x & -6 \\
 & & 6x^2 & +18x & \\
\hline
 & & & -19x & -6 \\
 & & & -19x & -57 \\
\hline
 & & & & 51
\end{array}
$$

Thus by dividing the given polynomial (1.5) with $x + 3$, we get the value of remainder as 51.

Example 1.4. Find the remainder when $p(x) = 3x^3 + x^2 + 2x + 5$ is divided by $x + 1$.

Solution. Given a polynomial

$$p(x) = 3x^3 + x^2 + 2x + 5 \tag{1.6}$$

and divisor is $x + 1$. Then by putting

$$x + 1 = 0 \quad \text{we get} \quad x = -1 \tag{1.7}$$

So, the remainder is calculated by substituting $x = -1$ from (1.7) into equation (1.6).

$$
\begin{aligned}
p(-1) =& 3(-1)^3 + (-1)^2 + (-1) + 5 \\
=& -3 + 1 - 2 + 5 \\
=& 1
\end{aligned}
$$

Thus, by dividing the given polynomial (1.6) with $x+1$, we get the value of remainder as 1.

Verification:

$$
\begin{array}{r|llll}
 & 3x^2 & -2x & +4 & \\
\hline
x+1 & 3x^3 & +x^2 & +2x & +5 \\
 & 3x^3 & +3x^2 & & \\
\hline
 & & -2x^2 & +2x & +5 \\
 & & -2x^2 & -2x & \\
\hline
 & & & 4x & +5 \\
 & & & 4x & +4 \\
\hline
 & & & & 1
\end{array}
$$

1.4.2 The Factor Theorem

The Factor theorem is a theorem which gives the relation between the factors and zeros of the polynomial. It states that

Theorem 1.2 (The Factor Theorem)**.** *A polynomial* $p(x)$ *has a factor* $(x-c)$ *if and only if* $p(c)=0$ *(i.e. c is the root of the polynomial* $p(x)$ *.*

Or

If c is the root of the polynomial equation $p(x)=0$ *i.e.* $p(c)=0$ *then* $(x-c)$ *is the factor of* $p(x)$*.*

Proof. Consider a polynomial $p(x)$ which has $x-c$ as one of its factor. Then from equation (1.3) in previous theorem 1.1 we have

$$p(x)=(x-c)\ q(x)+p(c) \tag{1.8}$$

Since $x-c$ is one of the factor of $p(x)$, therefore using equation (1.8) one can conclude that the value of remainder must be zero i.e. $p(c)=0$.

Converse: Since it is given that $p(c)=0$, therefore from equation (1.8) we have

$$p(x)=(x-c)\ q(x)$$

Thus $(x-c)$ is one of the factor of $p(x)$. □

Remark. Following statements are equivalent for any polynomial $p(x)$

1. The remainder is zero when $p(x)$ is exactly divided by $(x-c)$.
2. c is the solution of $p(x)$.
3. $(x-c)$ is a factor of $p(x)$.
4. c is the root of the polynomial $p(x)$ i.e. $p(c)=0$.

Remark. Factor theorem is a special case of the polynomial remainder theorem. i.e. when $r=0$ in equation (1.3), then $p(x)=(x-c)\ q(x)$

Example 1.5. Without actual division show that $2x^4-x^3-6x^2+4x-8$ is divisible by $x+2$ or $x+2$ is one of the factor of $2x^4-x^3-6x^2+4x-8$.

Solution. Consider the polynomial

$$p(x)=2x^4-x^3-6x^2+4x-8. \tag{1.9}$$

To show that $p(x)$ is divisible by $x+2$ or $x+2$ is one of the factor of $p(x)$, we have to show $p(-2) = 0$ (since $x + 2 = 0$ then $x = -2$). By putting $x = -2$ in the given polynomial (1.9), we get

$$\begin{aligned} p(-2) =& 2(-2)^4 - 2(-2)^3 - 6(-2)^2 + 4(-2) - 8 \\ =& 32 + 8 - 24 - 8 - 8 \\ =& 0 \end{aligned}$$

Since the value of remainder is zero, therefore $x+2$ is one of the factor of $2x^4 - x^3 - 6x^2 + 4x - 8$.

Example 1.6. Check whether $2x - 1$ is one of the factor of $2x^3 - x^2 - 2x + 1$ or $x = \frac{1}{2}$ is the zero of the polynomial equation $2x^3 - x^2 - 2x + 1 = 0$. Also verify the statement.

Solution. Since $2x - 1$ is the divisor of the polynomial

$$p(x) = 2x^3 - x^2 - 2x + 1, \tag{1.10}$$

then by putting

$$2x - 1 = 0 \quad \text{we get} \quad x = \frac{1}{2} \tag{1.11}$$

$$\begin{aligned} p\left(\frac{1}{2}\right) =& 2\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1 \\ =& \frac{1}{4} - \frac{1}{4} - 1 + 1 \\ =& 0 \end{aligned}$$

Hence $(2x - 1)$ is one of the factor of $2x^3 - x^2 - 2x + 1$ or $x = \frac{1}{2}$ is the zero/solution/root of the polynomial $2x^3 - x^2 - 2x + 1$.

Verification:

$$\begin{array}{r l} & x^2 \quad -1 \\ 2x-1 \big) & \overline{2x^3 - x^2 - 2x + 1} \\ & \underline{2x^3 - x^2} \\ & \qquad -2x + 1 \\ & \qquad \underline{-2x + 1} \\ & \qquad\qquad 0 \end{array}$$

Hence $(2x - 1)$ is one of the factor of $2x^3 - x^2 - 2x + 1$.

In-text Exercise 1.1. Solve the following questions:

(a) Without actual division find the remainder when $x^3 - 3x^2 + 6x - 5$ is divided by $x - 3$.

(b) Without actual division find the remainder when $4x^3 - 17x^2 + 9$ is divided by $3x - 5$.

(c) Find the value of k if $4x^3 - 2x^2 + kx + 5$ leaves remainder -10 when divided by $2x + 1$.

(d) Without actual division show that

  (i) $18x^{10} + 19x^5 + 1$ is divisible by $x + 1$.

  (ii) $r^3 - 1, r^4 - 1, r^5 - 1$ are divisible by $r - 1$.

(e) Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.

1.4.3 Synthetic Division

Synthetic Division is a shortcut way for the long Division. As we know long division with polynomials involve many steps, synthetic divisions carries the calculations even in a few steps
Here we learn synthetic divisions by linear polynomial. The advantage of this method over the remainder theorem and factor theorem is that using synthetic division we can find the quotient and remainder both of the given polynomial. Here we will understand the steps of synthetic division by taking an example.

Example 1.7. Consider a polynomial $p(x) = x^4 + 3x^3 - 2x - 5$ is divided by $x - 2$. What is the quotient and remainder in this case.

Step-1: Write the coefficients of the polynomial in descending order in a line by ignoring the powers of x. Also write the zero coefficient of missing powers of x.

Implication Since the coefficient of x^4 is 1, x^3 is 3, x^2 is 0, x is -2 and the value of constant is -5.

$*$	1	3	0	-2	-5

Step-2: Write the value of linear polynomial by which we have to divide the given polynomial. Here we have to divide $p(x)$ by $x - 2 = 0$ i.e. $x = 2$. The value of x is written at the place of $*$ in table of step-1.

Implication

2	1	3	0	-2	-5

Step-3: Write the same coefficient a_0 below in the third line

a	a_0	a_1	a_2	a_3	a_4
	a_0				

Implication

2	1	3	0	-2	-5
	1				

Step-4: Multiple a_0 by a and then write it below the entry a_1 in second row. Then add a_1 and a_0a and write it in the third row. Again multiply $a_1 + a_0a$ by a and then write it below the entry a_2. Then add a_2 and $a(a_0a + a_1)$ and write it in the third row and follow the same procedure for all the entries.

a	a_0	a_1 a_0a	a_2 $a(a_0a + a_1)$	a_3	a_4
	a_0	$a_0a + a_1$	$a(a_0a + a_1) +$ a_2	...	

Implication Multiply 2 by 1 and write it below 3.

2	1	3 2	0	-2	-5
	1				

add both the values and write it in the third row.

2	1	3 2	0	-2	-5
	1	5			

Again multiple 5 by 2 then write it below 0. Add both the number and write it in the third row.

2	1	3	0	-2	-5
		2	10		
	1	5	10		

By following the same procedure we get the following table

2	1	3	0	-2	-5
		2	10	20	36
	1	5	10	18	31

Remark.

1. Last entry in the third row represent the remainder of the polynomial after dividing it by a linear factor.
2. Quotient is given by multiplying the entry of third row with one less degree equation. For example, in the above question 31 is the remainder while quotient is $1.x^3 + 5.x^2 + 10.x + 18$.
3. We have to carry out all the steps of this synthetic division in a single table.

Example 1.8. Divide $x^3 + 3x^2 - 2x - 5$ by $x - 2$.

Solution. Here $x - 2 = 0$ implies $x = 2$. Hence

2	1	3	-2	-5
		2	10	16
	1	5	8	11

the remainder is 11 and the value of quotient is $1.x^2 + 5.x + 8 = x^2 + 5x + 8$.

Example 1.9. Divide $x^3 + x^2 + x + 1$ by $x + 1$

Solution. Here $x + 1 = 0$ implies $x = -1$. Hence

-1	1	1	1	1
		-1	0	-1
	1	0	1	0

the remainder is 0 and the value of quotient is $1.x^2 + 0.x + 1 = x^2 + 1$.

In-text Exercise 1.2. Solve the following questions:

(a) Divide $x^3 + 6x^2 + 10x - 1$ by $x - 3$.

(b) Without actual division find the remainder when $3x^3 - 17x^2 - x + 15$ is divided by $3x - 5$.

(c) Find the quotient of $2x^4 - x^3 - 6x^2 + 4x - 8$ by $x^2 - 4$.

1.4.4 Factored form of a polynomial

Polynomials can be written in the factored form. The factored form of a polynomial means it is written as a product of its factors.
Consider a polynomial of degree 'n' whose leading coefficients (coefficient of highest degree term) is not zero.

$$p(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n \quad (a_n \neq 0). \tag{1.12}$$

If $p(x) = 0,$ has the root α_1 (either real or imaginary), then by using factor theorem 1.2, $p(x)$ has one of the factor $(x - \alpha_1)$. So the polynomial equation (1.12) can be written as

$$p(x) \equiv (x - \alpha_1)g(x), \quad \text{where} \quad g(x) \equiv a_0x^{n-1} + a_1'x^{n-2} + ... + a_{n-1}'$$

If $g(x) = 0$ has the root α_2, then

$$g(x) \equiv (x - \alpha_2)g_1(x), \quad \text{where} \quad g_1(x) \equiv a_0x^{n-2} + b_1'x^{n-3} + ... + b_{n-2}'.$$

Therefore

$$p(x) \equiv (x - \alpha_1)(x - \alpha_2)g_1(x)$$

If $g_1(x) = 0$ has the root α_3, then continuing in the same way as above we get

$$p(x) \equiv a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)g_2(x)$$

Proceeding in the same way we get

$$p(x) \equiv a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)...(x - \alpha_n). \tag{1.13}$$

From the above discussion, we can conclude that if any equation $p(x) = 0$ of degree 'n' is known to have 'n' distinct roots α_1, $\alpha_2 ... \alpha_n$, then $p(x)$ can be expressed in the factored form (1.13).

Theorem 1.3. *An equation of degree 'n' cannot have more than 'n' distinct roots.*

Proof. Since a polynomial equation (1.12), $p(x) = 0$ of degree 'n' having 'n' roots α_1, $\alpha_2 ... \alpha_n$ can be written in the factored form as (1.13)

$$p(x) \equiv a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)...(x - \alpha_n). \tag{1.14}$$

Let equation (1.12) has one more root α different from $\alpha_1, \alpha_2...\alpha_n$ of polynomial equation $p(x)$. since α is the root of $p(x) = 0$, therefore $p(\alpha) = 0$. Hence from equation (1.14)

$$0 = p(\alpha) \equiv a_0(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)...(\alpha - \alpha_n). \tag{1.15}$$

Thus, one of the factor of right hand side of equation (1.15) must be zero say

$$\alpha - \alpha_i = 0$$
$$\alpha = \alpha_i$$

Hence any polynomial of degree 'n' cannot have more than 'n' distinct roots. □

1.4.5 Multiple Roots

Multiple roots of a polynomial are roots whose factors show up more than once in the complete factorization of the polynomial. The number of times a factor shows up in the complete factorization is called the multiplicity of the corresponding root.
Consider in equation (1.12), α_1 has multiplicity m_1 i.e. α_1 is the root repeating m_1 times. Also multiplicity of α_2 is m_2 and so on. Then equation (1.12) becomes

$$p(x) \equiv a_0(x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2}...(x - \alpha_k)^{m_k} \tag{1.16}$$

where $m_1 + m_2 + ... + m_k = n =$ degree of the polynomial. Also $\alpha_1, \alpha_2...\alpha_k$ are distinct roots of the polynomial.

Theorem 1.4. *An equation of degree n cannot have more than n roots, a root of multiplicity m being counted as m roots.*

Proof. Let α_1 has multiplicity m_1, α_2 has multiplicity m_2,.. and α_k has multiplicity m_k. Then

$$p(x) \equiv a_0(x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2}...(x - \alpha_k)^{m_k} \tag{1.17}$$

where $m_1 + m_2 + ... + m_k = n =$ degree of the polynomial. Also $\alpha_1, \alpha_2...\alpha_k$ are distinct roots of the polynomial. Since $(x - \alpha_1)^{m_1}$ is a factor of $p(x)$ or $p(x)$ is exactly divisible by $(x - \alpha_1)^{m_1}$, let us assume equation (1.17) has one more root α_1. Then multiplicity of α_1 becomes $m_1 + 1$. Thus

$$p(x) \equiv a_0(x - \alpha_1)^{m_1+1}(x - \alpha_2)^{m_2}...(x - \alpha_k)^{m_k} \tag{1.18}$$

Since, $p(x)$ is exactly divisible by $(x - \alpha_1)^{m_1}$, therefore it will not be exactly divisible by $(x - \alpha_1)^{m_1+1}$. (because α_1 is a root of multiplicity m_1.) An equation of degree n cannot have more than n roots, a root of multiplicity m being counted as m roots. □

Example 1.10. Let the polynomial equation $p(x) = 5(x - 2)(x - 6)^2(x - 3)^3(x - 4)^5$. Then 2 is the simple root, 6 is the root with multiplicity 2, 3 is the root with multiplicity 3 and 4 is the root with multiplicity 5 of given equation of degree 11 which has no further roots.

1.5 Fundamental Theorem of Algebra and its Consequences

Fundamental theorem of algebra also known as d'Alembert's theorem states that

Theorem 1.5. *Every non constant single variable polynomial or every algebric equation with complex coefficients has at least one complex (real and imaginary) roots*

or

Any polynomial of degree n has n roots.

Let us consider $p(x) = a_0x^n + a_1x^{n-1}... + a_n = 0$ be any equation of degree n where $a_0 \neq 0$. Since by using fundamental theorem of algebra, every equation of degree $n \geq 1$ has at least one root. Therefore using factored form of polynomial, it will have exactly n roots of the polynomial.

Remark. This theorem does not reveal what the roots of the polynomial equation are but it tells about the number of roots of an equation.

Example 1.11. How many total roots exist for the function $p(x) = -6x^2 + x^3 - 6 + 11x$.

Solution. Since the highest degree of the given polynomial is 3, therefore it will have exactly 3 roots. On solving $p(x) = -6x^2 + x^3 - 6 + 11x = 0$, we get the roots as $1, 2, 3$.

1.6 Theorems on imaginary, integral and rational roots

Theorem 1.6. *The complex roots occurs in conjugate pair.*

Or

In a given equation

$$a_0x^n + a_1x^{n-1} + ... + a_n = 0 \ \ and \ \ a_0 \neq 0 \tag{1.19}$$

where $a_0, a_1, ...a_n$ are coefficients of the equation and the value of all the coefficients are real constant number. If the given polynomial has a complex root $a + ib$ then it will also has the root $a - ib$ (i.e. its complex conjugate will also be the root of the equation.)

Proof. Proof of the theorem is omitted. □

Example 1.12. Find the roots of the polynomial equation $x^3 + x^2 + x + 1 = 0$.

Solution. The given equation is

$$p(x) = x^3 + x^2 + x + 1 = 0. \tag{1.20}$$

Since -1 satisfy the equation (1.20) ($p(-1) = -1 + 1 - 1 + 1 = 0$), therefore one of the root of equation (1.20) is -1. Thus $x + 1$ will be one of the factor of $p(x)$. Also the given equation is cubic, hence it will have three roots. Dividing the given polynomial by its factor we get the value of quotient. Rest two roots can be find by solving the equation of quotient. To find the quotient we will use the synthetic division.

-1	1	1	1	1
		-1	0	-1
	1	0	1	0

Hence the quotient is given as $q(x) = 1.x^2 + 0.x + 1 = x^2 + 1$. Thus the root of equation $x^2 + 1 = 0$ are found as $i, -i$. Hence we can conclude that roots occurs in pair. Thus if any polynomial whose all the coefficients are real and having one complex root then the conjugate of that root will also exist.

Remark. If the coefficients given in equation (1.19) are complex instead of real then complex root may or may not occurs in conjugate pair.

Example 1.13. Find the roots of the polynomial equation $x^2 - 7ix - 12 = 0$.

Solution. Since the given equation is

$$p(x) = x^2 - 7ix - 12 = 0. \tag{1.21}$$

Solution of this equation is given as

$$\begin{aligned} x =& \frac{7i \pm \sqrt{-49 + 48}}{2} \\ x =& \frac{7i \pm i}{2} \\ x =& 4i, 3i \end{aligned}$$

Here $4i$ and $3i$ are not conjugate of each other because the coefficients of the polynomial equation are not purely real.

Theorem 1.7. *The irrational roots occurs in conjugate pair.*

Or

In a given equation

$$a_0x^n + a_1x^{n-1} + ... + a_n = 0 \;\; and \;\; a_0 \neq 0 \tag{1.22}$$

where $a_0, a_1, ...a_n$ are coefficients of the equation and the value of all the coefficients are rational number. If the given polynomial has irrational root $a + \sqrt{b}$ then it will also has a pair of its root $a - \sqrt{b}$.

Proof. Proof of the theorem is omitted. □

Example 1.14. Find the roots of the polynomial equation $x^2 - 4x + 1 = 0$.

Solution. The given equation is

$$p(x) = x^2 - 4x + 1 = 0. \tag{1.23}$$

$$\begin{aligned} x =& \frac{4 \pm \sqrt{16-4}}{2} \\ x =& \frac{4 \pm 2\sqrt{3}}{2} \\ x =& 2 + \sqrt{3}, 2 - \sqrt{3}. \end{aligned}$$

Hence, we can conclude that roots occurs in pair $(2+\sqrt{3}, 2-\sqrt{3})$. Thus, if any polynomial whose all the coefficients are rational and having one irrational root then the conjugate of that root will also be the roots of that polynomial equation.

Remark. If the coefficients given in equation (1.22) are irrational instead of real then irrational root may or may not occurs in conjugate pair.

Example 1.15. Find the roots of the polynomial equation $x^2 - 3\sqrt{3}x + 6 = 0$.

Solution. Since the given equation is

$$p(x) = x^2 - 3\sqrt{3}x + 6 = 0. \tag{1.24}$$

Solution of this equation is given as

$$\begin{aligned} x =& \frac{3\sqrt{3} \pm \sqrt{27-24}}{2} \\ x =& \frac{3\sqrt{3} \pm \sqrt{3}}{2} \\ x =& 2\sqrt{3}, \sqrt{3} \end{aligned}$$

Here $2\sqrt{3}$ and $\sqrt{3}$ are not conjugate of each other because the coefficients of the polynomial equation are not purely rational.

Theorem 1.8. ***Theorem on integral roots**: Consider the equation of n degree*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} ... + a_n = 0, \tag{1.25}$$

whose all the coefficients $a_0, a_1, a_2, ...a_n$ are integers, then any integer root of that equation will be exact divisor of the constant term.

Proof. Let us assume that equation has an integer root 'x'. Then from the above equation (1.25)

$$\begin{aligned} -a_0x^n - a_1x^{n-1} - a_2x^{n-2} ... &= a_n, \qquad (1.26) \\ -x(a_0x^{n-1} + a_1x^{n-2} + a_2x^{n-3} ...) &= a_n, \end{aligned}$$

Since all the coefficients of the given equation are integers. Hence the quantity in parenthesis will be integer and constant term on the right hand side is also integer. Therefore x must be the complete or exact divisor of constant term. □

Remark. Converse of the above theorem may or may not be true i.e. any exact divisor of constant term may or may not be the root of the equation. For example, consider an equation $4x^2 + 4x + 1 = 0$. The exact divisor of constant term are ± 1. But equation have no integer roots i.e. both the roots are rational $\frac{-1}{2}, -\frac{1}{2}$.

Example 1.16. Find all the integral roots of the equation

$$x^3 + x^2 - 3x + 9 = 0. \tag{1.27}$$

Solution. Since all the coefficients of the given equation are integers, therefore first condition of the theorem is satisfied. Let us assume that equation has some integral roots. Then they must divide the constant term. So the exact divisor of constant term are $\pm 1, \pm 3, \pm 9$. Using synthetic division or factor theorem, we can check that ± 1 are not the roots of the equation.

-1	1	1	-3	9
		-1	-1	4
	1	0	-4	13

Since the last term of third row is not zero. Hence -1 is not root of this equation. Similarly we can check for the other roots also.

-3	1	1	-3	9
		-3	6	-9
	1	-2	3	0

Hence -3 is the integral root of this equation. Rest two roots can be obtained by solving the quotient equation i.e. $1.x^2 - 2.x + 3 = x^2 - 2x + 3 = 0$

Example 1.17. Find all the integral roots of the equation $x^4 + 4x^3 + 8x + 32 = 0$.

Solution. Since the constant term here has numerous exact divisors, therefore finding the roots will be quite difficult. To overcome this difficulty, we will transform this equation by taking a transformation $x = 2z$ into a equation whose constant term has less number of divisor. The transformed equation become

$$\begin{aligned} (2z)^4 + 4(2z)^3 + 8(2z) + 32 &= 0, \\ 16z^4 + 32z^3 + 16z + 32 &= 0, \\ z^4 + 2z^3 + z + 2 &= 0. \end{aligned} \tag{1.28}$$

In the transformed equation (1.28) constant term is 2, whose divisors are $\pm 1, \pm 2$. Now $+1, +2$ will not be the root of transformed equation (1.28) because all the coefficient here are positive. Now we will check for -1 and -2.

-1	1	2	0	1	2
		-1	-1	1	-2
	1	1	-1	2	0

-1	1	2	0	1	2
		-1	-1	1	-2
-2	1	1	-1	2	0
		-2	2	-2	
	1	-1	1	0	

Hence the two integral roots of the transformed equation (1.28) are $-1, -2$ and the quotient is $z^2-z+1 = 0$. By solving the quotient by quadratic formula, we can find the rest two roots of the equation (1.28). The integral roots of original equation are obtained by substituting these values in the transformation $x = 2z$ i.e. $x = 2.(-1), x = 2.(-2)$. Thus the integral roots of the original equation $x^4 + 4x^3 + 8x + 32 = 0$ are $-2, -4$.

1.6.1 Newton's method for integral roots

From equation (1.26), in the previous theorem 1.8, we obtain that a_n must be divisible by x or x must be complete divisor of a_n for a equation having integer coefficients. In equation (1.26), by taking a_n and a_{n-1} both on the right hand side, we can obtain one more condition

$$\begin{aligned} -a_0x^n - a_1x^{n-1} - a_2x^{n-2}... &= a_{n-1}x + a_n, \qquad (1.29)\\ -x^2(a_0x^{n-2} + a_1x^{n-3} + a_2x^{n-4}...) &= a_{n-1}x + a_n,\\ -x(a_0x^{n-1} + a_1x^{n-2} + a_2x^{n-3}...) &= a_{n-1} + \frac{a_n}{x}, \end{aligned}$$

in the similar way. Thus x^2 must be divisor of $a_{n-1}x+a_n$ or x must be divisor of $a_{n-1}+\frac{a_n}{x}$. Proceeding in the same way, we can obtained a set of conditions of divisibility which an integral roots must satisfy. i.e. x must be divisor of $a_{n-2} + \frac{a_{n-1}}{x} + \frac{a_n}{x^2}$. In last x must divide the final sum $a_0 + \frac{a_1}{x} + \frac{a_2}{x^2}....$ and the value of sum will be zero.

Remark.

1. x will be the root of any polynomial equation, if it satisfies all the conditions of Newton's method.

2. This method is quicker than synthetic division, as it detect the wrong choice at earlier steps and throws it out.

3. The value of final sum $a_0 + \frac{a_1}{x} + \frac{a_2}{x^2}....$ must be zero.

Example 1.18. Discuss the condition of Newton's method for example 1.16 to check the integral roots.

Solution. Since in example 1.16 we check that -3 is the root of the given equation (1.27), while 3 is not the root of the equation (1.27) using synthetic division. Here we will check these roots by the conditions of Newton's method. We will show that -3 satisfy all the conditions of Newton's method while 3 does not satisfies all.

$$\begin{aligned}
-3/9 &= -3\\
-3/-3+\frac{9}{-3} &= 2\\
-3/1+\frac{-3}{-3}+\frac{9}{(-3)^2} &= -1\\
-3/1+\frac{1}{-3}+\frac{-3}{(-3)^2}+\frac{9}{(-3)^3} &= 0
\end{aligned}$$

Thus -3 divided all the values completely. Hence it satisfy all the conditions of Newton's method. Thus -3 will be the root of equation (1.27). On the other side

$$\begin{aligned}
3/9 &= 3\\
3/-3+\frac{9}{3} &= 0\\
3\nmid 1+\frac{-3}{3}+\frac{9}{(3)^2}
\end{aligned}$$

Since 3 does not satisfies all the conditions of Newton's method. Hence 3 is not root of the given equation. In the similar way, we can check the conditions for all the roots.

1.6.2 Rational Roots

If the rational number $\frac{p}{q}$, a fraction in its lowest form (so that p and q are integers prime to each other, and $q \neq 0$) is the root of the given equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2}... + a_n = 0, \tag{1.30}$$

where all the coefficients $a_0, a_1, a_2, ...a_n$ are integers and $a_0 \neq 0$, then p is the divisor of a_n, while q is an divisor of a_0.

Remark. If in the equation (1.30), the value of coefficient a_0 is 1, then all the rational roots will be an integer.

Example 1.19. Find all the rational and integral roots of the equation

$$p(x) = 2x^3 - 9x^2 + 13x - 6 = 0 \tag{1.31}$$

Solution. Here $a_0 = 2$ and $a_n = -6$. Then for any rational root of the form $\frac{r}{s}$ r must be divisor of -6 while s must be divisor of 2. Since all the divisor of 2 are $\pm1, \pm2$ and all the divisor of -6 are $\pm1, \pm2, \pm3, \pm6$. Hence all the possible number of fractional roots of the given equation (1.31) are $\frac{\pm 1}{\pm 2}, \frac{\pm 3}{\pm 2}$ ie $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$. By substituting all the values one by one in equation (1.31), we can find the rational roots of the given equation.

$$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + 13\left(\frac{1}{2}\right) - 6 = -\frac{3}{2} \neq 0$$
$$p\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 9\left(-\frac{1}{2}\right)^2 + 13\left(-\frac{1}{2}\right) - 6 = -15 \neq 0$$
$$p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 13\left(\frac{3}{2}\right) - 6 = 0$$
$$p\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 - 9\left(-\frac{3}{2}\right)^2 + 13\left(-\frac{3}{2}\right) - 6 = -\frac{210}{4} \neq 0$$

Thus, one integral root of the given equation is $\frac{3}{2}$ and rest roots can be found by using integral theorem 1.8 and Newton's method 1.6.1.

Remark. Some important results on roots of an equation

1. Every equation of an odd degree has at least one real roots of a sign opposite to that of its last term.
2. Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other one negative.

In-text Exercise 1.3. Find all the integral and rational roots of the following equations

(a) $x^3 + 8x^2 + 13x + 6 = 0$

(b) $x^3 + 12x^2 - 32x - 256 = 0$

(c) $2x^3 + 3x^2 - 11x - 6 = 0$

(d) $32x^3 - 6x - 1 = 0.$

(e) Find the equation of the lowest degree with real coefficients having $1 + i$ and $2 - i$ as two of its roots

1.7 Descartes' rule of signs

1.7.1 Descartes' rule of signs-Positive roots

In algebra, Descartes' rule of signs is used for finding the maximum possible number of positive real roots of a polynomial without actually solving it. It state that
The number of positive root of an equation $p(x) = 0$ can not exceed the number of changes of sign from (+ to −) or from (− to +) in the terms occurring in $p(x)$.

Or

The possible number of the positive root of an equation $p(x) = 0$ is equal to the number of sign changes in the coefficients of the terms or less than the sign changes by a multiple of 2.

Remark. 1. This rule does not give the exact number of roots of the polynomial. Also, it does not identify the roots of the polynomial.

2. Before applying the Descartes' rule of signs make sure to arrange the terms of the polynomial in descending order. For example it should be in the order ..., x^5, x^4, x^3, x^2, x and constant term.

3. While counting the sign change, do not write the terms that have a coefficient to be 0. For example $3x^2 - 1$ will not be written as $3x^2 + 0x - 1$.

Example 1.20. Find the possible number of positive real roots of the polynomial $x^3+3x^2-x-x^4-2$.

Solution. Since the terms of the polynomial are not in the descending order of exponents, therefore we will first make the terms in descending order of exponent.

$$p(x) = -x^4 + x^3 + 3x^2 - x - 2 \tag{1.32}$$

Now, we will count the number of sign changes in the given polynomial (1.32).

$$\begin{aligned}
&\text{from} - x^4 \text{ to } + x^3 \text{ there is one sign change}(-\text{to}+)\\
&\text{from} + x^3 \text{ to } + 3x^2 \text{ there is no sign change}(+\text{to}+)\\
&\text{from} + 3x^2 \text{ to } - x \text{ there is one sign change}(+\text{to}-)\\
&\text{from} - x \text{ to } - 2 \text{ there is no sign change}(-\text{to}-)
\end{aligned}$$

Thus there are two sign changes in the given polynomial $p(x)$ and hence possible number of positive real roots of the polynomial is 2 or 0.

1.7.2 Descartes' rule of signs-Negative roots

In a polynomial equation $p(x) = 0$, if x is replaced by $-x$, then the resulting equation will have the same roots as the original except that their signs will be changed. Thus the negative roots of $p(x) = 0$ are the positive roots of $p(-x) = 0$. Hence the Descartes's rule of signs for negative roots is stated as
The number of negative real roots of an equation $p(x) = 0$ can not exceed the number of changes of sign from ($+$ to $-$) or from ($-$ to $+$) in the terms occurring in $p(-x)$.

Or

The possible number of the negative roots of the polynomial $p(x)$ is equal to the number of sign changes in the coefficients of the terms of $p(-x)$ or less than the sign changes by a multiple of 2.

Remark. 1. This rule does not give the exact number of roots of the polynomial. Also, it does not identify the roots of the polynomial.

2. Before applying the Descartes' rule of signs make sure to arrange the terms of the polynomial in descending order of exponents.

Example 1.21. Find the possible number of real roots of the polynomial $x^3 - x^2 - 14x + 24$ and verify.

Solution. Since the terms of the polynomial are already in the descending order of exponents, therefore we will count the number of sign changes in the given polynomial

$$p(x) = +x^3 - x^2 - 14x + 24. \tag{1.33}$$

$$\begin{aligned}
&\text{from } + x^3 \text{ to } - x^2 \text{ there is one sign change}(+\text{to}-)\\
&\text{from } - x^2 \text{ to } - 14x \text{ there is no sign change}(-\text{to}-)\\
&\text{from } - 14x \text{ to } + 24 \text{ there is one sign change}(-\text{to}+)
\end{aligned}$$

Thus there are two sign changes in the given polynomial $p(x)$ and hence possible number of positive real roots of the polynomial is 2 or 0. Similarly for finding the number of negative roots, we will check the sign change in the given polynomial by replacing x by $-x$. Thus by taking $(-x)$ in place of x, required polynomial becomes

$$\begin{aligned}
p(-x) =& (-x)^3 - (-x)^2 - 14(-x) + 24\\
=& - x^3 - x^2 + 14x + 24
\end{aligned} \tag{1.34}$$

$$\begin{aligned}
&\text{from } - x^3 \text{ to } - x^2 \text{ there is no sign change}(-\text{to}-)\\
&\text{from } - x^2 \text{ to } + 14x \text{ there is one sign change}(-\text{to}+)\\
&\text{from } + 14x \text{ to } + 24 \text{ there is no sign change}(+\text{to}+)
\end{aligned}$$

Thus there are one sign changes in the given polynomial $p(-x)$ and hence possible number of negative real roots of the polynomial $p(x)$ is 1.

Verification: Since the polynomial given in equation (1.33) is cubic, therefore we can find the roots of this equation. The roots of the equation are obtained as $2, 3, -4$. Thus we can see that there are two positive roots and one negative root of the given polynomial (1.33).

Example 1.22. Determine the possible number of real solutions of the polynomial $4x^7 + 3x^6 + x^5 + 2x^4 - x^3 + 9x^2 + x + 1 = 0$.

Solution. For the possible positive roots, we will check the sign changes in the given polynomial

$$p(x) = +4x^7 + 3x^6 + x^5 + 2x^4 - x^3 + 9x^2 + x + 1. \tag{1.35}$$

$$\begin{aligned}
&\text{from } + 2x^4 \text{ to } - x^3 \text{ there is one sign change}(+\text{to}-)\\
&\text{from } - x^3 \text{ to } + 9x^2 \text{ there is one sign change}(-\text{to}+)
\end{aligned}$$

Thus there are two sign changes in the given polynomial $p(x)$ and hence possible number of positive real solution of the polynomial is 2 or 0. For finding the possible number of negative solution, we will check the sign change in the given polynomial by replacing x by $-x$. Thus by taking $(-x)$ in place of x, required polynomial becomes

$$\begin{aligned} p(-x) &= +4(-x)^7 + 3(-x)^6 + (-x)^5 + 2(-x)^4 - (-x)^3 + 9(-x)^2 + (-x) + 1. \\ &= -4x^7 + 3x^6 - x^5 + 2x^4 + x^3 + 9x^2 - x + 1 \end{aligned} \tag{1.36}$$

from $-4x^7$ to $+3x^6$ there is one sign change($-$to$+$)
from $+3x^6$ to $-x^5$ there is one sign change($+$to$-$)
from $-x^5$ to $+2x^4$ there is one sign change($-$to$+$)
from $+9x^2$ to $-x$ there is one sign change($+$to$-$)
from $-x$ to $+1$ there is one sign change($-$to$+$)

Since there are five sign changes in the given polynomial $p(-x)$ therefore possible number of negative real solutions of the polynomial $p(x)$ is 5, 3 or 1.

Thus there are two or zero positive solutions, and five, three or one negative solutions of given polynomial (1.35).

1.7.3 Application of Descartes' rule of signs for finding the Imaginary roots

In a polynomial equation $p(x) = 0$, if the sum of number of maximum possible positive roots and maximum possible negative roots are less then the degree of the polynomial then there will be existence of imaginary roots in that polynomial. It is also clear that for the real coefficients, the imaginary roots occur in conjugate pair.
The number of imaginary roots in a given polynomial equation $p(x) = 0$ is either equal to $(n - p - q)$ or greater then $(n - p - q)$ by an even number where n is the degree of the polynomial while p and q are the positive and negative roots of the polynomial respectively.

Remark. This application of Descartes's rule of signs for finding the imaginary roots is possible only in case of incomplete equation. For the complete equation total number of sign variation in $p(x)$ and $p(-x)$ will be equal to degree of the polynomial.(A polynomial is called incomplete if the coefficient of some terms are zero. For example $x^4 + x - 1$ is incomplete polynomial as the coefficient of x^3 and x^2 term is zero while $x^4 - x^3 + x^2 - x + 1$ is a complete polynomial.)

Example 1.23. Prove that the equation $3x^7 - x^4 + x^3 - 1 = 0$, has at least four imaginary roots.

Solution. Let the equation be $p(x) = 0$, where

$$p(x) = +3x^7 - x^4 + x^3 - 1. \tag{1.37}$$

The number of sign change in $p(x)$ is

$$\begin{aligned}&\text{from} + 3x^7 \text{ to } - x^4 \text{ there is one sign change}(+\text{to}-)\\&\text{from} - x^4 \text{ to } + x^3 \text{ there is one sign change}(-\text{to}+)\\&\text{from} + x^3 \text{ to } - 1 \text{ there is one sign change}(+\text{to}-)\end{aligned}$$

Thus there are three changes of sign and hence possible number of positive real roots of the polynomial are 3 or 1. The number of sign change in $p(-x)$ is

$$p(-x) = -3x^7 - x^4 - x^3 - 1$$

Since all the terms of $p(-x)$ are negative, hence there is no sign change for $p(-x)$. Thus there is no real negative roots of the given polynomial. Now we will construct a table with all possibilities. Note that the degree of the given polynomial (1.37) is 7.

Number of positive real roots	Number of negative real roots	Number of imaginary roots
3	0	$7-(3+0)=4$
1	0	$7-(1+0)=6$

Hence the total number of imaginary roots of equation (1.37) are either 4 or 6.

In-text Exercise 1.4. Solve the following questions:

(a) Find the possible number of positive, negative and imaginary roots of the polynomial equation $x^3 - x^2 + x - 1$.

(b) Find the nature of the roots of the equation $x^4 + 15x^2 + 7x - 11 = 0$.

(c) Show that the equation $x^n + 1 = 0$, has no real roots when n is even and -1 is the only root when n is odd.

(d) Find the number of imaginary roots of the equation $x^4 - 3x^2 - x + 1 = 0$

1.8 Summary

In the end of the chapter, we know

1. Polynomials can be written in the factored form.
2. Using synthetic division, any polynomial can be divided by a linear polynomial without performing a long calculations.
3. Imaginary and irrational roots occurs in pair for the real and rational coefficients respectively.
4. How to find the number of positive, negative and imaginary roots of an polynomial equation without solving it.

1.9 Self-Assessment Exercises

1. Check whether the polynomial $p(x) = 4x^3 + 4x^2 - x - 1$ is a multiple of $2x + 1$.

2. For what value of k is the polynomial $p(x) = 2x^3 - kx^2 + 3x + 10$ exactly divisible by $x - 2$.

3. If two polynomials $2x^3 + ax^2 + 4x - 12$ and $x^3 + x^2 - 2x + a$ leave the same remainder when divided by $x - 3$, find the value of a and also find the remainder.

4. Find the quotient of $x^3 - 5x^2 - 2x + 24$ by $x - 4$, and then divide the quotient by $x - 3$. What are the roots of $x^3 - 5x^2 - 2x + 24 = 0$.

5. If two roots of the equation $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$ are 1 and -3, then find the remaining two roots.

6. Find the equation of the lowest degree with rational coefficients having $3 - \sqrt{5}$ and $5 + \sqrt{2}$ as two of its roots.

7. Find all the integral and rational roots of $6y^3 - 11y^2 + 6y - 1 = 0$.

8. Find all the integral roots of $x^4 + 4x^3 + 8x + 32 = 0$ by using Newton's method for integral roots.

9. Find all the rational roots of $x^3 - \frac{1}{2}x^2 - 2x + 1 = 0$.

10. Find the nature of roots of the given equation
$$x^n + 1 = 0$$
when n is either odd or even.

11. Find all possible values of imaginary roots for the given polynomial $p(x) = x^3 + 3x^2 - x - x^5 + 7$.

12. Find the superior limit of the number of imaginary roots of the equation
$$5x^8 - 6x^3 + x^2 + 1 = 0.$$

1.10 Solutions to In-text Exercises

Exercise 1.1

(a) By substituting $x = 3$ in polynomial $p(x) = x^3 - 3x^2 + 6x - 5$, we get $p(3) = 13$, which is the value of remainder.

(b) same as above

(c) By substituting $x = -\frac{1}{2}$ in $p(x) = 4x^3 - 2x^2 + kx + 5$, we get $4 - \frac{k}{2}$ which is given to be equal to -10. Thus $k = 28$.

(d) By substituting $x = -1$ in polynomial $p(x) = 18x^{10} + 9x^5 + 1$, we get $p(-1) = 0$.

Exercise 1.2

(a) This can be done by synthetic division as

3	1	6	10	-1
		3	27	111
	1	9	37	110

the remainder is 110 and the value of quotient is $1.x^2 + 9.x + 37 = x^2 + 9x + 37$.

(b) This can be done by synthetic division as

$\frac{5}{3}$	3	-17	-1	15
		5	-20	-35
	3	-12	-21	-20

the remainder is -20 and the value of quotient is $3.x^2 - 12.x - 21 = 3x^2 - 12x - 21$.

(c) $x^2 - 4 = 0$ is $(x - 2)(x + 2) = 0$

2	2	-1	-6	4	-8
		4	6	0	8
	2	3	0	4	0

and

-2	2	3	0	4
		-4	2	-4
	2	-1	2	0

The value of quotient is $2.x^2 - 1.x + 2 = 2x^2 - x + 2$.

Exercise 1.3

(a) Since $x^3 + 8x^2 + 13x + 6$ has the coefficient of highest degree as 1. Therefore all the rational roots will be integers. Also factor/divisor of 6 are $\pm1, \pm2, \pm3, \pm6$. Now all the terms of the given equation are positive. Thus there will be no positive roots of the given equation. Hence total choices of roots are $-1, -2, -3, -6$. Using Newton's method we can check that -2 & -3 does not satisfy Newton's condition. Thus all the roots of given equation are $-1, -1, -6$, where -1 is the root with multiplicity 2.

(b) Since the constant term here has numerous exact divisors. Therefore we will transform this equation into a simpler equation by taking a transformation $x = 4z$.

(c) All possible roots of the given equation $2x^3 + 3x^2 - 11x - 6$ are ±1, ±2, ±3, ±6, $\frac{\pm1}{\pm2}$, $\frac{\pm3}{\pm2}$. Using Newton's method we can check that 2 & -3 satisfy Newton's all condition. Also $p\left(-\frac{1}{2}\right) = 0$. Thus, all the roots of given equation are $2, -3, -\frac{1}{2}$.

(d) All possible roots of the given equation $32x^3 - 6x - 1$ are $\pm1, \frac{\pm1}{\pm2}, \frac{\pm1}{\pm4}, \frac{\pm1}{\pm8}, \frac{\pm1}{\pm16}, \frac{\pm1}{\pm32}$.

(e) Since the coefficient of the required equation are given to be real. Therefore in that case complex roots occurs in pair. Thus the required equation must have at least four roots as $1 + i$, $1 - i$, $2 - i$ and $2 + i$. The equation is

$$\begin{aligned}(x - (1 - i))(x - (1 + i))(x - (2 + i))(x - (2 - i)) = 0\\ ((x - 1) + i)((x - 1) - i))((x - 2) - i)((x - 2) + i) = 0\\ ((x - 1)^2 + 1)((x - 2)^2 + 1) = 0.\end{aligned}$$

is the required equation.

Exercise 1.4

(a) Since total number of sign change in $p(x) = 0$ are 3. Hence total number of positive roots are $3, 1$. Also, total number of sign change in $p(-x) = 0$ are 0. Hence total number of negative roots are 0.

Number of positive real roots	Number of negative real roots	Number of imaginary roots
3	0	$3-(3+0) = 0$
1	0	$3-(1+0) = 2$

Hence the total number of imaginary roots of given polynomial equation are either 0 or 2.

(b) Since there are one sign change in case of $p(x)$ and one sign change in case of $p(-x)$. Thus there are one positive and one negative roots and $4 - (1 + 1) = 2$ imaginary roots.

(c) When n is even then there is no sign change in $p(x)$ and $p(-x)$. Hence no real root with exist. Also in case of odd power of n, there is no sign change in $p(x)$, while one sign change in case of $p(-x)$. Thus only one negative root will exist which will be -1.

(d) Since there are two sign change in $p(x)$ and two sign change in $p(-x)$. Hence

Number of positive real roots	Number of negative real roots	Number of imaginary roots
2	2	$4-(2+2)=0$
0	0	$4-(0+0)=4$
2	0	$4-(2+0)=2$
0	2	$4-(0+2)=2$

The inferior limit to the imaginary roots of given polynomial equation is 0.

Suggested Readings

1. Burnside, W.S., & Panton, A.W. (1979). The Theory of Equations. Vol. 1. Eleventh Edition, (Fourth Indian Reprint. S. Chand & Co. New Delhi), Dover Publications, Inc.

2. Dickson, Leonard Eugene (2009). First Course in the Theory of Equations. John Wiley & Sons, Inc. The Project Gutenberg eBook (http://www.gutenberg.org/ebooks/29785).

Lesson - 2

Relation Between the Roots and Coefficients of Equations

Structure

2.1 Learning Objectives

Students will be able to

- recognize the relation between the roots and coefficients of a quadratic and cubic equation.
- understand how the relation between the roots and coefficients of a quadratic and cubic equation can be extended to n degree polynomial equation.
- find the roots of a higher order polynomial equation when relation between its roots are given.

2.2 Introduction

Roots of polynomials are solutions for given polynomials where the function is equal to zero. When its comes to polynomials, roots becomes particularly important. They allow us to break down our polynomial equation into simpler terms that we can understand and solve

more easily. In the previous chapter, we had learned a lot of theorems based on roots of the polynomials. In this chapter, we will learn the relation between roots and coefficients of a polynomial equation. Using this relation, we will be able to find the roots of a polynomial equation of degree n.
A polynomial equation of degree n

$$p(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2}...a_{n-1}x + a_n = 0, \tag{2.1}$$

has exactly n roots. Let the n roots of equation (2.1) are $\alpha_1, \alpha_2, ..., \alpha_n$, then

$$p(x) \equiv a_0(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n). \tag{2.2}$$

2.3 Relation Between Roots and Coefficients of Equations

Here we will learn relation between roots and coefficients of a quadratic and cubic equation first. Later this result will be extended for any polynomial of degree n.

Relation between roots and coefficients of an quadratic equation

Consider a quadratic equation

$$ax^2 + bx + c = 0, \tag{2.3}$$

whose roots are given as α_1 and α_2. Roots of the quadratic equations (2.3) can be found using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.4}$$

Therefore, the roots of equation (2.3) are found as $\alpha_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\alpha_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Now, the sum of the roots is

$$\begin{aligned}\alpha_1 + \alpha_2 =& \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ =& -\frac{2b}{2a} = -\frac{b}{a} = -\frac{\text{coefficient of} x}{\text{coefficient of} x^2}.\end{aligned}$$

and the product of the roots is

$$\begin{aligned}\alpha_1.\alpha_2 =& \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \cdot \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) \\ =& \left(-\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 \\ =& \frac{4ac}{(2a)^2} \\ =& \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of} x^2}.\end{aligned}$$

Thus $\alpha_1 + \alpha_2 = -\frac{\text{coefficient of} x}{\text{coefficient of} x^2}$ **and** $\alpha_1.\alpha_2 = \frac{\text{constant term}}{\text{coefficient of} x^2}$ **represent the required relations between roots (i.e.** α_1 **and** α_2**) and coefficients (i.e.** a, b **and** c**) of the equation** $ax^2 + bx + c = 0$.

Example 2.1. Find the zeros/roots of the polynomial $p(x) = x^2 + 7x + 12$ and verify the relation between its zeros and coefficients.

Solution. The roots of the given equation are obtained as

$$\begin{aligned} x^2 + 7x + 12 &= 0, \qquad (2.5)\\ x^2 + 3x + 4x + 12 &= 0,\\ (x+3)(x+4) &= 0,\\ x &= -3, -4. \end{aligned}$$

Thus, the roots of the equation (2.5) are $\alpha_1 = -3, \alpha_2 = -4$.

Verification:

$$\text{Sum of the roots } (\alpha_1 + \alpha_2) = -3 + (-4) = -\frac{\text{coefficient of} x}{\text{coefficient of} x^2} = -\frac{7}{1} = -7$$

$$\text{Product of the roots } (\alpha_1.\alpha_2) = -3.(-4) = \frac{\text{constant term}}{\text{coefficient of} x^2} = \frac{12}{1} = 12.$$

Example 2.2. If α and β are the roots of the equation $3x^2 - 5x + 2 = 0$, then find the value of

1. $\alpha^2 + \beta^2$
2. $\alpha^3 + \beta^3$
3. $\frac{1}{\alpha} + \frac{1}{\beta}$

Solution. The given equation is

$$3x^2 - 5x + 2 = 0. \qquad (2.6)$$

Since, the roots of equation (2.6) are α and β, therefore using the relation between roots and coefficients, we get

$$\begin{aligned} \alpha + \beta &= -\frac{\text{coefficient of} x}{\text{coefficient of} x^2} = -\left(\frac{-5}{3}\right) = \frac{5}{3}\\ \alpha.\beta &= \frac{\text{contant term}}{\text{coefficient of} x^2} = \frac{2}{3} \qquad (2.7) \end{aligned}$$

1. Now

$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta\\ &= \left(\frac{5}{3}\right)^2 - 2.\left(\frac{2}{3}\right)\\ &= \frac{13}{9}. \end{aligned}$$

2. We know

$$\begin{aligned}\alpha^3+\beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\\ &= \left(\frac{5}{3}\right)^3 - 3.\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\\ &= \frac{35}{27}.\end{aligned}$$

3. Also,

$$\begin{aligned}\frac{1}{\alpha}+\frac{1}{\beta} &= \frac{(\alpha+\beta)}{\alpha\beta}\\ &= \left(\frac{5/3}{2/3}\right)\\ &= \frac{5}{2}.\end{aligned}$$

Relation between roots and coefficients of an cubic equation

Consider a cubic equation

$$ax^3+bx^2+cx+d=0, \quad (a\neq 0) \tag{2.8}$$

whose roots are given as α_1, α_2 and α_3. Therefore, by equation (2.1) and (2.2)

$$\begin{aligned}ax^3+bx^2+cx+d &\equiv a(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\\ x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a} &\equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\\ x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a} &\equiv x^3-(\alpha_1+\alpha_2+\alpha_3)x^2+(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)x-\alpha_1\alpha_2\alpha_3.\end{aligned}$$

On comparing the same power's of x, we get the following relations

$$\begin{aligned}\alpha_1+\alpha_2+\alpha_3 = \sum_{i=1}^{3}\alpha_i &= -\frac{b}{a} = -\frac{\text{coefficient of} x^2}{\text{coefficient of} x^3}\\ \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3 = \sum_{i,j=1}^{3}\alpha_i\alpha_j &= \frac{c}{a} = \frac{\text{coefficient of} x}{\text{coefficient of} x^3}\\ \alpha_1\alpha_2\alpha_3 = \prod_{i=1}^{3}\alpha_i &= -\frac{d}{a} = -\frac{\text{constant term}}{\text{coefficient of} x^3}\end{aligned}$$

Thus $\alpha_1+\alpha_2+\alpha_3 = -\frac{\text{coefficient of} x^2}{\text{coefficient of} x^3}$**,** $\alpha_1.\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3 = \frac{\text{coefficient of x}}{\text{coefficient of} x^3}$**, and** $\alpha_1.\alpha_2.\alpha_3 = -\frac{\text{constant term}}{\text{coefficient of} x^3}$ **represent the required relations between roots (i.e.** α_1**,** α_2 **and** α_3**) and coefficients (i.e.** a, b, c **and** d**) of the equation** $ax^3+bx^2+cx+d=0$.

Example 2.3. Find the equation whose roots are given as $3, 3, -2$. Also, verify the relation between roots and coefficients.

Solution. Given three roots are $\alpha_1 = 3, \alpha_2 = 3, \alpha_3 = -2$. The required equation is given as

$$\begin{aligned}(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) &= 0\\ (x-3)(x-3)(x-(-2)) &= 0\\ x^3-4x^2-3x+18 &= 0 \qquad (2.9)\end{aligned}$$

Verification:

$$\text{Sum of the roots } (\alpha_1+\alpha_2+\alpha_3) = 3+3-2 = -\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3} = -\left(\frac{-4}{1}\right) = 4.$$

$$\begin{aligned}\text{Sum of the product of two roots } (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) &= 3.3+3.(-2)+3.(-2)\\ &= \frac{\text{coefficient of } x}{\text{coefficient of } x^3} = \left(\frac{-3}{1}\right) = -3.\end{aligned}$$

$$\text{Product of the roots } (\alpha_1.\alpha_2.\alpha_3) = 3.3.(-2) = -\frac{\text{constant term}}{\text{coefficient of } x^3} = -\left(\frac{18}{1}\right) = -18.$$

In-text Exercise 2.1. Solve the following questions:

(a) If the difference between roots of the equation $x^2-13x+k=0$ is 17. Find k

(b) If α and β are the roots of the equation $3x^2+7x-2=0$, find the values of

  (i) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

  (ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Relation Between Roots and Coefficients of n degree equation

Consider a polynomial equation of degree n

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0, \qquad (2.10)$$

whose roots are given by $\alpha_1, \alpha_2, \alpha_3..., \alpha_n$. Therefore by using equation (2.1) and (2.2)

$$\begin{aligned}a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n &\equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)...(x-\alpha_n)\\ x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_n}{a_0} &\equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)...(x-\alpha_n)\\ x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_n}{a_0} &\equiv \left(x^2-(\alpha_1+\alpha_2)x+\alpha_1\alpha_2\right)(x-\alpha_3)...(x-\alpha_n)\\ x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_n}{a_0} &\equiv x^n - (\alpha_1+\alpha_2+\alpha_3+...+\alpha_n)x^{n-1}\\ &\quad +(\alpha_1\alpha_2+\alpha_1\alpha_3+...+\alpha_2\alpha_3+...)x^{n-2} + ...(-1)^n\alpha_1\alpha_2\alpha_3...\alpha_n.\end{aligned}$$

On comparing the same power's of x, we get the following relations

$$S_1 = \alpha_1 + \alpha_2 + \alpha_3 + ...\alpha_n = \sum_{i=1}^{n} \alpha_i = -\frac{a_1}{a_0} = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + ... + \alpha_2\alpha_1 + ... = \sum_{i,j=1}^{n} \alpha_i\alpha_j = \frac{a_2}{a_0} = \frac{\text{coefficient of } x^{n-2}}{\text{coefficient of } x^n}$$

$$\vdots$$

$$S_n = \alpha_1\alpha_2\alpha_3...\alpha_n = \prod_{i=1}^{n} \alpha_i = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$$

Here, S_k denotes the sum of the products of the roots taken k at a time. For example S_2 denotes the sum of the product of the roots taken 2 at a time.

Remark. If in the equation (2.10), coefficient of the highest term is unity, i.e.

$$x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0, \tag{2.11}$$

then, the above relation become quite easy to apply. In that case sum of the roots become negative times of coefficient of x^{n-1} while sum of the product of the two roots becomes coefficient of x^{n-2} and same as follows with alternate sign change. i.e.

$$S_1 = \alpha_1 + \alpha_2 + \alpha_3 + ...\alpha_n = -\text{ coefficient of } x^{n-1} = -a_1$$

$$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + ... + \alpha_2\alpha_1 + ... = \text{coefficient of } x^{n-2} = a_2$$

$$\vdots$$

$$S_n = \alpha_1\alpha_2\alpha_3...\alpha_n = (-1)^n\text{constant term} = (-1)^n a_n$$

Remark. Some special roots of a cubic equation

1. If three roots of a cubic equation are given in A.P. (arithmetic progression), then roots may be chosen as $a - d, a, a + d$. The choice of particular form of these roots make the calculation easier to solve. As in this case

$$\begin{aligned}
\text{Sum of the roots} &= a - d + a + a + d = 3a \\
\text{Sum of product of the two roots} &= (a - d).a + a.(a + d) + (a + d).(a - d) \\
&= a^2 - ad + a^2 + ad + a^2 - d^2 \\
&= 3a^2 - d^2 \\
\text{product of the roots} &= (a - d).a.(a + d) \\
&= a(a^2 - d^2).
\end{aligned}$$

2. If three roots of a cubic equation are given in G.P. (geometric progression), then roots

may be chosen as $\frac{a}{r}, a, ar$. In this case

$$\begin{aligned}\text{Sum of the roots} &= \frac{a}{r} + a + ar \\ \text{Sum of product of the two roots} &= \frac{a}{r}.a + a.ar + ar.\frac{a}{r} \\ &= \frac{a^2}{r} + a^2 r + a^2 \\ \text{product of the roots} &= \frac{a}{r}.a.ar \\ &= a^3\end{aligned}$$

3. If three roots of a cubic equation are given in H.P. (harmonic progression), then roots may be chosen as α, β, γ where $\beta = \frac{2\alpha\gamma}{\alpha+\gamma}$.
   Note: A harmonic progression is a progression which are formed by taking the reciprocals of an arithmetic progression i.e. if α, β, γ are in H.P. then $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ will be in A.P. Therefore, we have

$$\begin{aligned}\frac{1}{\beta} - \frac{1}{\alpha} &= \frac{1}{\gamma} - \frac{1}{\beta} \\ \frac{2}{\beta} &= \frac{\alpha+\gamma}{\alpha\gamma} \\ \beta &= \frac{2\alpha\gamma}{\alpha+\gamma}\end{aligned}$$

Remark. Some special roots of a bi-quadratic equation

1. If four roots of a bi-quadratic equation are given in A.P. (arithmetic progression), then roots may be chosen as $a - 3d, a - d, a + d, a + 3d$.

2. If four roots of a bi-quadratic equation are given in G.P. (geometric progression), then roots may be chosen as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

3. If four roots of a bi-quadratic equation are given in H.P. (harmonic progression), then roots may be chosen as $\alpha, \beta, \gamma, \delta$ where $\beta = \frac{2\alpha\gamma}{\alpha+\gamma}, \gamma = \frac{2\beta\delta}{\beta+\delta}$.

2.4 Applications to solution of Equations

We are sometimes required to find conditions under which the roots of given equations are related. Also, sometimes we have to solve a equation when a relation between its roots are given or roots are related by some relation. Following examples will illustrate the procedure for solving such types of problems.

Example 2.4. Solve the equation

$$x^3 - 9x^2 + 23x - 16 = 0, \tag{2.12}$$

whose roots are in Arithmetic progression (A.P.)

Solution. Let the roots be chosen as $a-d, a, a+d$. Therefore, we get

$$\text{Sum of the roots} = a-d+a+a+d = 3a = -\frac{\text{coefficient of} x^2}{\text{coefficient of} x^3} = 9$$
$$3a = 9$$
$$a = 3.$$

Also $$(a-d).a + a.(a+d) + (a-d).(a+d) = 3a^2 - d^2 = \frac{\text{coefficient of} x}{\text{coefficient of} x^3} = 23$$
$$3a^2 - d^2 = 23$$
$$3.(3)^2 - d^2 = 23$$
$$d^2 = 4$$
$$d = \pm 2$$

Thus by taking $a = 3, d = 2$, roots become $a-d = 1, a = 3, a+d = 5$ i.e. $1, 3, 5$.

Note: By taking $a = 3, d = -2$, we get the same set of roots as $a-d = 5, a = 3, a+d = 1$ i.e. $5, 3, 1$.

Example 2.5. Solve the equation

$$2x^3 - x^2 - 22x - 24 = 0 \tag{2.13}$$

two of its roots being in the ration $3:4$.

Solution. Since the roots are given in the ratio $3:4$, therefore let us assume the roots of the equation (2.13) as $3\alpha, 4\alpha, \beta$. Thus we have

$$3\alpha + 4\alpha + \beta = \frac{1}{2}$$
$$7\alpha + \beta = \frac{1}{2} \tag{2.14}$$
$$3\alpha.4\alpha + 4\alpha.\beta + 3\alpha.\beta = -\frac{22}{2} = -11$$
$$12\alpha^2 + 7\alpha\beta = -11 \tag{2.15}$$
$$3\alpha.4\alpha.\beta = -\left(\frac{-24}{2}\right) = 12$$
$$12\alpha^2\beta = 12$$
$$\alpha^2\beta = 1 \tag{2.16}$$

From eqution (2.14) and (2.15), we get

$$12\alpha^2 + 7\alpha\left(\frac{1}{2} - 7\alpha\right) = -11$$
$$-74\alpha^2 + 7\alpha + 22 = 0$$
$$\alpha = \frac{7 \pm \sqrt{49 + 6512}}{2.(74)}$$
$$\alpha = -\frac{1}{2}, \frac{22}{37}. \tag{2.17}$$

Since $\alpha = \frac{22}{37}$ does not satisfy the equation (2.13). Hence by putting the value of $\alpha = -\frac{1}{2}$ in equation (2.16), the value of $\beta = 4$. Thus three roots of equation (2.13) are given as $-\frac{3}{2}, -2, 4$.

Example 2.6. Solve the equation

$$3x^3 - 26x^2 + 52x - 24 = 0, \tag{2.18}$$

whose roots are in G.P.

Solution. Let the roots in G.P. are $\frac{a}{r}, a, ar$. We know,

$$\frac{a}{r} + a + ar = -\left(\frac{-26}{3}\right) = \frac{26}{3} \tag{2.19}$$

$$\frac{a}{r}.a + a.ar + \frac{a}{r}.ar = \frac{52}{3}$$

$$\frac{a^2}{r} + a^2 r + a^2 = \frac{52}{3} \tag{2.20}$$

$$\frac{a}{r}.a.ar = -\left(\frac{-24}{3}\right) = 8$$

$$a^3 = 8 \tag{2.21}$$

Dividing equation (2.19) by equation (2.20), we get

$$\frac{\frac{a}{r} + a + ar}{a(\frac{a}{r} + ar + a)} = \frac{\frac{26}{3}}{\frac{52}{3}}$$

$$\frac{1}{a} = \frac{1}{2}$$

$$a = 2.$$

Putting the value of 'a' in equation (2.19), we get

$$\frac{2}{r} + 2 + 2r = \frac{26}{3}$$

$$6r^2 - 20r + 6 = 0$$

$$3r^2 - 10r + 3 = 0$$

$$3r^2 - 9r - r + 3 = 0$$

$$(3r - 1)(r - 3) = 0$$

$$r = \frac{1}{3}, 3$$

Taking $a = 2$ and $r = 3$, roots of the equation (2.18) are obtained as $\frac{2}{3}, 2, 6$.

Note:- By taking $a = 2$ and $r = \frac{1}{3}$, we get the same sets of roots.

Example 2.7. Solve the equation $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$, whose roots are given in A.P.

Solution. Let the roots be chosen as $a-3d, a-d, a+d, a+3d$. Then by the relation between roots and coefficients, we get

$$\begin{aligned} a-3d+a-d+a+d+a+3d &= -2 \\ 4a &= -2 \\ a &= -\frac{1}{2}. \end{aligned}$$

Also,

$$\begin{aligned} (a-3d)(a-d)+(a-d)(a+d)+(a+d)(a+3d)+(a+3d)(a-3d)+(a-3d)(a+d) \\ +(a+3d)(a-d) &= -21 \\ 6a^2-10d^2 &= -21 \\ 6\left(\frac{-1}{2}\right)^2+21 &= 10d^2 \qquad \left[\because \left(a=-\frac{1}{2}\right)\right] \\ d^2 &= \frac{9}{4} \\ d &= \pm\frac{3}{2}. \end{aligned}$$

Taking $a=-\frac{1}{2}$ and $d=\frac{3}{2}$, four roots of given equation are obtained as

$$\begin{aligned} a-3d &= -\left(\frac{1}{2}\right)-3\left(\frac{3}{2}\right) = -5 \\ a-d &= -\left(\frac{1}{2}\right)-\left(\frac{3}{2}\right) = -2 \\ a+d &= -\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right) = 1 \\ a+3d &= -\left(\frac{1}{2}\right)+3\left(\frac{3}{2}\right) = 4 \end{aligned}$$

Thus the four roots of given equation are $-5, -2, 1, 4$.

Note: By taking $a=-\frac{1}{2}$ and $d=-\frac{3}{2}$, we will get the same set of roots.

Example 2.8. Solve the cubic equation

$$2x^3-9x^2+12x-4=0, \tag{2.22}$$

given that two of its roots are equal.

Solution. Let the three roots of the equation are α, α, β (since two roots are equal). Then we have

$$\alpha+\alpha+\beta = 2\alpha+\beta = \frac{9}{2} \tag{2.23}$$

$$\alpha.\alpha+\alpha.\beta+\beta.\alpha = \alpha^2+2\alpha\beta = \frac{12}{2} = 6 \tag{2.24}$$

$$\alpha.\alpha.\beta = \alpha^2\beta = -\left(\frac{-4}{2}\right) = 2 \tag{2.25}$$

From equation (2.23), $\beta = \frac{9}{2} - 2\alpha$. Putting the value of β in equation (2.24), we get

$$\begin{aligned} \alpha^2 + 2\alpha\left(\frac{9}{2} - 2\alpha\right) &= 6 \\ -3\alpha^2 + 9\alpha - 6 &= 0 \\ \alpha^2 - 3\alpha + 2 &= 0 \\ \alpha &= 1, 2. \end{aligned}$$

Since $\alpha = 1$ does not satisfy the given equation (2.22). Therefore by taking $\alpha = 2$, we get $\beta = \frac{1}{2}$. Thus, three roots of the given equation (2.22) are found as $2, 2, \frac{1}{2}$.

Example 2.9. Solve the equation

$$3x^3 + 11x^2 + 12x + 4 = 0, \tag{2.26}$$

whose roots are given in Harmonic progression (H.P.)

Solution. Let the three roots in harmonic progression of the given equation are α, β, γ where $\beta = \frac{2\alpha\gamma}{\alpha+\gamma}$. Since

$$\alpha + \beta + \gamma = -\frac{11}{3} \tag{2.27}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{12}{3} = 4 \tag{2.28}$$

$$\alpha\beta\gamma = -\frac{4}{3}. \tag{2.29}$$

Also,

$$\begin{aligned} \beta &= \frac{2\alpha\gamma}{\alpha + \gamma} \\ \alpha\beta + \beta\gamma &= 2\alpha\gamma \end{aligned}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = 2\alpha\gamma + \alpha\gamma = 3\alpha\gamma \tag{2.30}$$

From equation (2.28) and (2.30), we get

$$\begin{aligned} 3\alpha\gamma &= 4 \\ \alpha\gamma &= \frac{4}{3}. \end{aligned}$$

Putting the value of $\alpha\gamma$ in equation (2.29), we get

$$\begin{aligned} \frac{4}{3}.\beta &= -\frac{4}{3} \\ \beta &= -1. \end{aligned}$$

Thus, one of the root of the equation (2.26) is obtained as $\beta = -1$. Therefore, by factor theorem 1.2 $(x + 1)$ will be one of the factor of equation (2.26). Thus dividing the equation (2.26) by $(x + 1)$, we get

-1	3	11	12	4
		-3	-8	-4
	3	8	4	0

Hence the reduced quadratic equation is obtained as $3x^2 + 8x + 4 = 0$. By solving this equation we get

$$x = \frac{-8 \pm \sqrt{64 - 48}}{6} = \frac{-8 \pm 4}{6} = \frac{-2}{3}, -2. \tag{2.31}$$

Thus all the required roots of the equation (2.26) are $-1, -2, -2/3$.

Example 2.10. Find the necessary condition for the roots of the equation

$$x^3 - px^2 + qx - r = 0, \tag{2.32}$$

to be in

(i) A.P.

(i) G.P.

(i) H.P.

Solution. To find the necessary condition for the roots to be in A.P., G.P., and H.P., we have to find a relation between coefficients of the equation by using relation between the roots.

(i) Let the roots in A.P. are $a - d, a, a + d$, then we know

$$a - d + a + a + d = -(-p) = p$$
$$3a = p$$
$$a = \frac{p}{3} \tag{2.33}$$
$$(a - d).a + a.(a + d) + (a - d)(a + d) = q$$
$$3a^2 - d^2 = q \tag{2.34}$$
$$(a - d).a.(a + d) = r$$
$$a^3 - ad^2 = r \tag{2.35}$$

Since a is one of the root of equation (2.32). Therefore it will satisfy the equation. Hence we get

$$a^3 - pa^2 + qa - r = 0. \tag{2.36}$$

Putting the value of a from equation (2.33) in equation (2.36), we get

$$\left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0$$
$$\left(\frac{p^3}{27}\right) - \left(\frac{p^3}{9}\right) + \left(\frac{pq}{3}\right) - r = 0$$
$$2p^3 - 9pq + 27r = 0 \tag{2.37}$$

is the required relation/condition for the roots to be in A.P.

Note: This relation can also be found by solving equation (2.33) and (2.34) for a and d and putting the value of a and d in equation (2.35).

(i) Let the roots in G.P. are $\frac{a}{d}, a, ad$, then we know

$$\frac{a}{d} + a + ad = -(-p) = p \tag{2.38}$$

$$\frac{a}{d}.a + a.(ad) + \frac{a}{d}(ad) = q$$

$$\frac{a^2}{d} + a^2d + a^2 = q \tag{2.39}$$

$$\frac{a}{d}.a.(ad) = r$$

$$a^3 = r$$

$$a = (r)^{\frac{1}{3}} \tag{2.40}$$

Since a is one of the root of equation (2.32). Therefore it will satisfy the equation. Hence, we get

$$a^3 - pa^2 + qa - r = 0. \tag{2.41}$$

Putting the value of a from equation (2.40) in equation (2.41), we get

$$\left(r^{\frac{1}{3}}\right)^3 - p\left(r^{\frac{1}{3}}\right)^2 + q\left(r^{\frac{1}{3}}\right) - r = 0$$

$$r - \left(pr^{\frac{2}{3}}\right) + \left(qr^{\frac{1}{3}}\right) - r = 0$$

$$p^3r - q^3 = 0 \tag{2.42}$$

is the required relation/condition for the roots to be in G.P.

(i) Let the roots in H.P. are α, β, γ, where $\beta = \frac{2\alpha\gamma}{\alpha+\gamma}$. We know

$$\alpha + \beta + \gamma = -(-p) = p \tag{2.43}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = q \tag{2.44}$$

$$\alpha\beta\gamma = r \tag{2.45}$$

Also, $$\beta = \frac{2\alpha\gamma}{\alpha+\gamma}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = 3\alpha\gamma. \tag{2.46}$$

From equation (2.46) and (2.44), we get the value of $\alpha\gamma = \frac{q}{3}$. Putting the value of $\alpha\gamma$ in equation (2.45), we get the value of $\beta = \frac{3r}{q}$. Since β is one of the root of equation (2.32), therefore it will satisfy the equation. Hence, we get

$$\beta^3 - p\beta^2 + q\beta - r = 0. \tag{2.47}$$

Putting the value of β in equation (2.47), we get

$$\left(\frac{3r}{q}\right)^3 - p\left(\frac{3r}{q}\right)^2 + q\left(\frac{3r}{q}\right) - r = 0$$
$$\left(\frac{27r^3}{q^3}\right) - p\left(\frac{9r^2p}{q}\right) + 3r - r = 0$$
$$27r^2 - 9pq^2r + 2q^3 = 0 \tag{2.48}$$

is the required relation/condition for the roots to be in H.P.

Example 2.11. The product of the two roots of the equation

$$x^4 + x^3 - 16x^2 - 4x + 48 = 0 \tag{2.49}$$

is 6. Find all the roots of the given equation.

Solution. Let four roots of the given equation (2.49) are $\alpha, \beta, \gamma, \delta$. Therefore by using relation between roots and coefficients we get

$$\alpha + \beta + \gamma + \delta = -1 \tag{2.50}$$
$$\alpha\beta + \beta\gamma + \alpha\gamma + \beta\delta + \alpha\delta + \gamma\delta = -16$$
$$\alpha\beta + (\alpha + \beta)(\gamma + \delta) + \gamma\delta = -16 \tag{2.51}$$
$$\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = 4$$
$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 4 \tag{2.52}$$
$$\alpha\beta\gamma\delta = 48. \tag{2.53}$$

Also, it is given that $\alpha\beta = 6$. Therefore from equation (2.53), $\gamma\delta = 8$. Putting the value of $\alpha\beta$ and $\gamma\delta$ in equation (2.51) and (2.52), we get

$$6 + (\alpha + \beta)(\gamma + \delta) + 8 = -16$$
$$(\alpha + \beta)(\gamma + \delta) = -30 \tag{2.54}$$
$$6(\gamma + \delta) + 8(\alpha + \beta) = 4 \tag{2.55}$$

Let $\alpha + \beta = l$ and $\gamma + \delta = m$ Then

$$lm = -30$$
$$6m + 8l = 4 \tag{2.56}$$

Substituting the value of $m = -\frac{30}{l}$ in equation (2.56), we get

$$8l + 6\left(-\frac{30}{l}\right) = 4$$
$$2l^2 - l - 45 = 0$$
$$2l^2 - 10l + 9l - 45 = 0$$
$$(2l + 9)(l - 5) = 0$$
$$l = 5, -\frac{9}{2} \tag{2.57}$$

Taking $l = 5$, we get $m = -6$. Thus the value of $\alpha + \beta = 5$ and $\gamma + \delta = -6$ (Note: $\alpha + \beta = -\frac{9}{2}$ and $\gamma + \delta = \frac{60}{9}$ does not satisfy equation (2.50).)
By taking $\alpha + \beta = 5$ and $\alpha\beta = 6$, we get

$$\begin{aligned} \alpha + \frac{6}{\alpha} &= 5 \\ \alpha^2 - 5\alpha + 6 &= 0 \\ \alpha &= 2, 3 \end{aligned} \tag{2.58}$$

Taking $\alpha = 2$, we get $\beta = 3$ and taking $\alpha = 3$, we get $\beta = 2$. Also by taking $\gamma + \delta = -6$ and $\gamma\delta = 8$, we get

$$\begin{aligned} \gamma + \frac{8}{\gamma} &= -6 \\ \gamma^2 + 6\gamma + 8 &= 0 \\ \gamma &= -4, -2 \end{aligned} \tag{2.59}$$

Taking $\gamma = -4$, we get $\delta = -2$ and taking $\delta = -2$, we get $\gamma = -4$. Thus the four roots of equation (2.49) are given as $2, 3, -4, -2$.

In-text Exercise 2.2. Solve the following questions:

(a) Solve the equation $8x^3 - 14x^2 + 7x - 1 = 0$, whose roots are in G.P.

(b) Solve the equation $x^3 - 5x^2 - 16x + 80 = 0$, the sum of two of its roots being zero.

(c) Solve the equation $x^3 - 9x^2 + 23x - 15 = 0$, whose two roots are in the ratio of $3 : 5$.

(d) Solve the equation $2x^3 + x^2 - 7x - 6 = 0$, given that the difference of two of its roots is 3.

(e) Solve the equation $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$, sum of two of its roots being equal to the sum of the other two.

2.5 Summary

In the end of the chapter, we know

1. The relation between roots and coefficients of an polynomial equation.

2. Using relation between roots and coefficients how to find relation between roots of a polynomial equation.

3. Using relation between roots, how to find all the roots of the given equation.

2.6 Self Assessment Exercise

1. Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$, whose roots are in A.P.

2. Determine the roots of the equation
$$2x^3 - 4x^2 - 2x + 4 = 0,$$
whose two roots are equal in magnitude and opposite in sign.

3. The roots of the equation $3x^3 - x^2 - 3x + 1 = 0$, are in H.P. Find the roots.

4. If the roots of the equation
$$x^3 - px^2 + qx - r = 0,$$
be in harmonic progression, show that the mean root is $\frac{3r}{q}$.

5. If the sum of two roots of the equation
$$4x^4 - 24x^3 + 31x^2 + 6x - 8 = 0,$$
be zero, find all the roots of the equation.

6. Solve the equation
$$x^4 + 15x^3 + 70x^2 + 120x + 64 = 0,$$
whose roots are in geometric progression (G.P.).

7. Solve the equation
$$x^3 - 5x^2 - 2x + 24 = 0,$$
given that the product of two of its roots is 12.

8. Find a necessary condition for the roots of the equation $ax^3 + bx^2 + cx + d = 0$ to be in A.P.

9. The equation
$$3x^4 - 25x^3 + 50x^2 - 50x + 12 = 0,$$
has two roots whose product is 2. Find all the roots.

10. Solve the equation
$$2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0,$$
given that the product of two roots is equal to the product of the other two roots.

11. Solve the equation
$$x^3 - 9x^2 + 14x + 24 = 0,$$
two of whose roots are in the ratio of 3 : 2.

12. Find the condition for a cubic equation to have a pair of roots.

2.7 Solutions to In-text Exercises

Exercise 2.1

(a) Since $\alpha+\beta=13$, and $\alpha-\beta=17$. From both the equation, we can find the value of α and β. Hence, by using $\alpha\beta=k$, we can obtain the value of k.

(b) Since $\alpha+\beta=-\frac{7}{3}$ and $\alpha\beta=-\frac{2}{3}$, therefore

(i) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=-\frac{61}{6}$

(ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{67}{9}$.

Exercise 2.2

(a) Three roots in G.P. are taken as $\frac{a}{r}, a, ar$. Using relation between coefficient and roots of a cubic equation, we get $a=\frac{1}{2}, r=\frac{1}{2}$. Hence the three roots are $1, \frac{1}{2}, \frac{1}{4}$.

(b) Let the three roots of the given equation are α, β, γ, where $\alpha+\beta=0$. Since

$$\alpha+\beta+\gamma=5$$
$$\therefore \gamma=5.$$

Also

$$\alpha\beta\gamma=-80$$
$$\alpha.(-\alpha).5=-80$$
$$\alpha=\pm 4$$

Hence the roots of the equation are $4, -4, 5$.

(c) Let the three roots are $3\alpha, 5\alpha, \beta$. Then

$$3\alpha+5\alpha+\beta=9 \tag{2.60}$$
$$15\alpha^2+8\alpha\beta=23 \tag{2.61}$$
$$15\alpha^2\beta=15 \tag{2.62}$$

On solving equation (2.60) and (2.61), we get $\alpha=1$ or $\alpha=\frac{23}{49}$. $\alpha=\frac{23}{49}$ does not satisfied the equation. Hence by taking $\alpha=1$, we get $\beta=1$, thus the roots are obtained as $3, 5\&1$.

(d) Let α, β, γ be the three roots of the given equation. Since $\alpha-\beta=3, \alpha+\beta+\gamma=-\frac{1}{2}, \alpha\beta\gamma=3$. On solving we get $\alpha=2, \beta=-1, \gamma=-\frac{3}{2}$.

(e) Let the four roots are $\alpha, \beta, \gamma, \delta$. Then

$$\alpha+\beta=\gamma+\delta \tag{2.63}$$
$$\alpha+\beta+\gamma+\delta=-2 \tag{2.64}$$
$$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=-21 \tag{2.65}$$
$$(\alpha+\beta)\gamma\delta+(\gamma+\delta)\alpha\beta=22 \tag{2.66}$$
$$\alpha\beta\gamma\delta=40 \tag{2.67}$$

On solving equation (2.63) and (2.64) we get $\alpha + \beta = \gamma + \delta = -1$ Also using equation (2.65) $\alpha\beta + \gamma\delta = -22$. Solving it with other equation, we get $\alpha = 1, \beta = -2, \gamma = 4, \delta = -5$.

Suggested Readings

1. Burnside, W.S., & Panton, A.W. (1979). The Theory of Equations. Vol. 1. Eleventh Edition, (Fourth Indian Reprint. S. Chand & Co. New Delhi), Dover Publications, Inc.

2. Dickson, Leonard Eugene (2009). First Course in the Theory of Equations. John Wiley & Sons, Inc. The Project Gutenberg eBook (http://www.gutenberg.org/ebooks/29785).

Lesson - 3

De Moivre's Theorem and its Applications

Structure

3.1 Learning Objectives

Students will be able to learn

- De Moivre's theorem for integer and rational index.
- application of fractional index De Moivre's theorem for finding the roots of a complex number.
- n^{th} root of unity and its properties

3.2 Introduction

In order to compute powers of complex numbers, we must consider the process of repeated multiplication. The process of repeated multiplication give rise a pattern. This pattern is the core of the theorem named after the French mathematician Abraham De Moivre. De Moivre's theorem gives a formula for computing powers of complex numbers. In this chapter we will explain the main concept of De- Moivre's theorems for integer and rational index. Also n^{th} roots of unity and their properties along with some examples are explained in details.

3.3 Complex Numbers

Complex numbers are the numbers that are expressed in the form of $z = a + ib$ where $a, b \in \mathbb{R}$. Here 'i' is the imaginary number called 'iota' and its value is $\sqrt{-1}$. Complex numbers has two parts, a which is called the real part and is denoted by Re(z) and another is b called the imaginary part and is denoted by Im(z). If the Im(z)=0, then we say that the complex number z is purely real and if the Re(z)=0, then we say that the complex number is purely imaginary.

Example 3.1. $2 + 3i, 5 + 7i, -9 + 12i, -6 - 8i$ are the examples of complex numbers

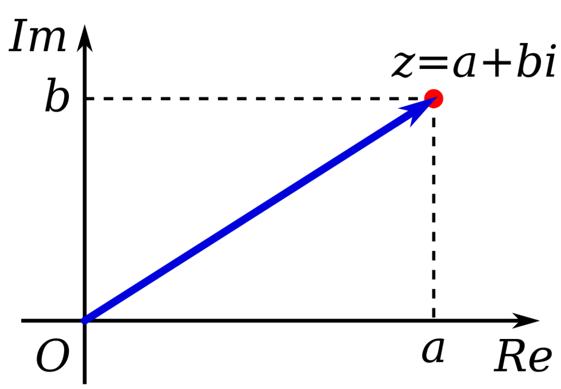

Figure 3.1: Graph of a complex number in argand plane.

This diagram represent how a complex number is represented geometrically in the 2-d plane.

Conjugate of a complex number

A complex conjugate of a complex number is another complex number that has the same real part as the original complex number and the imaginary part has the same magnitude but opposite sign. Conjugate of a complex number $z = a + ib$ is $a - ib$ which is denoted by $\bar{z}$.

Power of i

Since $i = \sqrt{-1}$

$$\begin{aligned} i^2 &= -1 \\ i^3 &= i^2.i = -i \\ i^4 &= i^2.i^2 = -1. - 1 = 1 \\ i^{-1} &= \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = -i \end{aligned}$$

Algebra of complex numbers

Let's understand the different algebra of complex number one by one

Equality of complex number

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are two complex numbers. Then $z_1 = z_2$ if and only if

$$\begin{aligned} Re(z_1) &= Re(z_2) \\ Im(z_1) &= Im(z_2). \end{aligned}$$

Addition of complex numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are two complex numbers, then sum of two complex number is calculated as

$$\begin{aligned} z_1 + z_2 &= (a_1 + ib_1) + (a_2 + ib_2) \\ &= (a_1 + a_2) + i(b_1 + b_2) \end{aligned}$$

Difference of complex numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are two complex numbers, then difference of two complex number is calculated as

$$\begin{aligned} z_1 - z_2 &= (a_1 + ib_1) - (a_2 + ib_2) \\ &= (a_1 - a_2) + i(b_1 - b_2) \end{aligned}$$

Multiplication of complex numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are two complex numbers, then multiplication of two complex number is calculated as

$$\begin{aligned} z_1 \times z_2 &= (a_1 + ib_1) \times (a_2 + ib_2) \\ &= (a_1a_2 - b_1b_2) + i(a_2b_1 + a_1b_2) \end{aligned}$$

Example 3.2. Let $z_1 = 1 + 7i$ and $z_2 = 4 + 5i$, then $z_1 \times z_2$?

Solution. $z_1 \times z_2$= $(1 + 7i) \times (4 + 5i) = 4 + 28i + 5i + 35i^2 = 4 + 33i - 35 = 33i - 31.$

Division of complex numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are two complex numbers, then division of two complex number is calculated as

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{(a_1 + ib_1)}{(a_2 + ib_2)} \\
\frac{z_1}{z_2} &= \frac{(a_1 + ib_1)}{(a_2 + ib_2)} \times \frac{(a_2 - ib_2)}{(a_2 - ib_2)} \qquad \text{(On rationalization)} \\
\frac{z_1}{z_2} &= \frac{(a_1 a_2 + b_1 b_2)}{(a_2^2 + b_2^2)} + \frac{i(a_2 b_1 - a_1 b_2)}{(a_2^2 + b_2^2)}.
\end{aligned}$$

Example 3.3. Express $\frac{3+2i}{4-i}$ in the form of $x + iy$.

Solution. Here

$$\begin{aligned}
\frac{3 + 2i}{4 - i} &= \frac{(3 + 2i)(4 + i)}{(4 - i)(4 + i)} \\
&= \frac{12 - 2 + i(8 + 3)}{16 + 1} \\
&= \frac{10 + 11i}{17} \\
&= \frac{10}{17} + i\frac{11}{17}.
\end{aligned}$$

Example 3.4. Find the value of $(1 + 2i)^3$

Solution.

$$\begin{aligned}
(1 + 2i)^3 &= (1 + (2i)^3 + 3.2i + 3.2i.2i) \\
&= (1 - 8i + 6i - 12) \\
&= (-11 - 2i)
\end{aligned}$$

Polar form of a complex number

Polar form of a complex number is another way to represent a complex number. In this form, we find the real and imaginary components in terms of r and θ, where r is the length of the vector and θ is the angle made with the positive direction of x axis. Let

$$x + iy = r(\cos\theta + i\sin\theta)$$

Equating the real and imaginary parts, we get

$$x = r\cos\theta \tag{3.1}$$
$$y = r\sin\theta \tag{3.2}$$

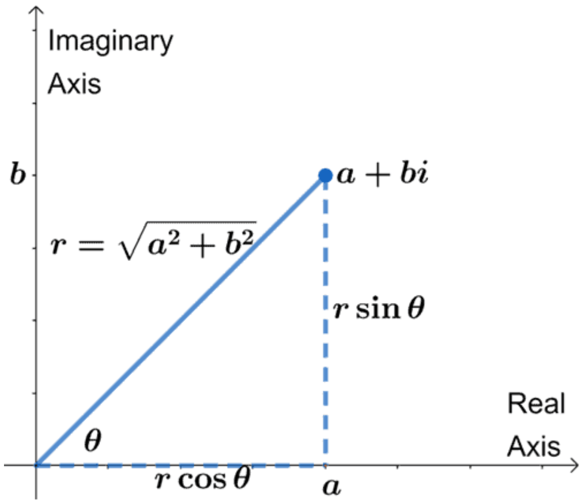

Figure 3.2: Graph of polar form of a complex number.

By squaring and adding equation (3.1) and (3.2), we get

$$x^2+y^2 = r^2\sin^2\theta + r^2\cos^2\theta$$
$$x^2+y^2 = r^2$$
$$r = \sqrt{x^2+y^2}$$

Here we take the positive value of r, as r denotes the length of the vector. Also dividing equation (3.2) by equation (3.1), we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \tag{3.3}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)\ (x \neq 0) \tag{3.4}$$

$r=\sqrt{x^2+y^2}$ is called the modulus of the complex number while $\theta=\tan^{-1}\left(\frac{y}{x}\right)$ is called the amplitude or arguments of the complex number.
Note: Since there are many values of θ which satisfies the equation $\theta=\tan^{-1}\left(\frac{y}{x}\right)$. Therefore, the value which lies between $-\pi<\theta\leq\pi$ is called the principal value of the amplitude. We shall generally take the principal value of θ.

Example 3.5. Express $\sqrt{3}-i$ in the polar form or standard form.

Solution. Let

$$\sqrt{3}-i = r(\cos\theta + i\sin\theta)$$

Equating the real and imaginary parts, we get

$$\sqrt{3} = r\cos\theta \tag{3.5}$$
$$-1 = r\sin\theta \tag{3.6}$$

By squaring and adding equation (3.5) and (3.6), we get

$$
\begin{aligned}
(\sqrt{3})^2 + (-1)^2 &= r^2 \sin^2\theta + r^2\cos^2\theta \\
4 &= r^2 \\
r &= 2
\end{aligned}
$$

Also dividing equation (3.6) by equation (3.5), we get

$$
\begin{aligned}
\frac{-1}{\sqrt{3}} &= \frac{r\sin\theta}{r\cos\theta} = \tan\theta \\
\theta &= \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) \qquad (3.7) \\
\theta &= -\frac{\pi}{6}
\end{aligned}
$$

Thus, the modulus of the complex number is 2 and argument is $-\frac{\pi}{6}$ and the polar form is written as $\sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$.

Example 3.6. Express $\sin\phi + i\cos\phi$ in the polar form or standard form.

Solution. Let

$$\sin\phi + i\cos\phi = r(\cos\theta + i\sin\theta)$$

Equating the real and imaginary parts, we get

$$
\begin{aligned}
\sin\phi &= r\cos\theta \qquad (3.8) \\
\cos\phi &= r\sin\theta \qquad (3.9)
\end{aligned}
$$

By squaring and adding equation (3.8) and (3.9), we get

$$
\begin{aligned}
(\sin\phi)^2 + (\cos\phi)^2 &= r^2\sin^2\theta + r^2\cos^2\theta \\
1 &= r^2 \\
r &= 1
\end{aligned}
$$

Also dividing equation (3.9) by equation (3.8), we get

$$
\begin{aligned}
\frac{\cos\phi}{\sin\phi} &= \frac{r\sin\theta}{r\cos\theta} = \tan\theta \\
\theta &= \tan^{-1}(\cot\phi) \qquad (3.10) \\
\theta &= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \phi\right)\right) \\
\theta &= \frac{\pi}{2} - \phi
\end{aligned}
$$

Thus, the modulus of the complex number is 1 and argument is $\frac{\pi}{2} - \phi$ and the polar form is written as $\sin\phi + i\cos\phi = \left(\cos\left(\frac{\pi}{2} - \phi\right) + i\sin\left(\frac{\pi}{2} - \phi\right)\right)$.

In-text Exercise 3.1. Solve the following

(a) Find the roots of the quadratic equation $x^2 - 6x + 10 = 0$.

(b) Express the following complex number in the polar form

(i) $1 + i\sqrt{3}$

(ii) $-2 + 2i$

(iii) $1 - i\sqrt{3}$

(iv) $5 + 2i$

(c) Express the complex number $\frac{2+3i}{1+i}$ in the form of $a + ib$. Find its modulus and amplitude.

(d) Show that $|\cos\theta + i\sin\theta| = 1$

3.4 De Moivre's Theorem

Theorem 3.1 (De Moivre's theorem for integral index)**.** *It states that the integral power of a complex number in polar form is equal to the product of the same power of modulus with multiplication of the argument by the same power.*
i.e.

$$(\cos\theta + \iota\sin\theta)^n = (\cos n\theta + \iota\sin n\theta), \quad \textit{where n is any integer}$$

or

$$(r\cos\theta + \iota r\sin\theta)^n = r^n(\cos n\theta + \iota\sin n\theta)$$

or

$$(e^{\iota\theta})^n = e^{in\theta}$$

Proof. **Case-1 If n is a positive integer.**
We will prove the theorem by principle of mathematical induction.

$$\text{For } n = 1: (r\cos\theta + ir\sin\theta)^1 = r^1(\cos\theta + i\sin\theta) \tag{3.11}$$

$$\begin{aligned}\text{For } n = 2: \ (r\cos\theta + ir\sin\theta)^2 =& (r\cos\theta + ir\sin\theta) * (r\cos\theta + ir\sin\theta)\\ =& (r^2\cos^2\theta + i^2r^2\sin^2\theta + 2ir^2\cos\theta\sin\theta)\\ =& r^2(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta) \ \ (\because i^2 = -1)\\ =& r^2(\cos 2\theta + i\sin 2\theta).\end{aligned}$$

Let us assume that the result is true for $n = k$, i.e.

$$\text{For } n = k: (r\cos\theta + ir\sin\theta)^k = r^k(\cos k\theta + i\sin k\theta) \tag{3.12}$$

We will show that the result is true for $n = k + 1$

$$\begin{aligned}
(r\cos\theta + ir\sin\theta)^{k+1} =& (r\cos\theta + ir\sin\theta)^k * (r\cos\theta + ir\sin\theta)\\
=& r^k(\cos k\theta + i\sin k\theta)(r\cos\theta + ir\sin\theta)(\text{by using equation (3.12)})\\
=& r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\
=& r^{k+1}(\cos k\theta\cos\theta - \sin k\theta\sin\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta)\\
=& r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)(\text{Using trigonometric identities})
\end{aligned}$$

Hence, the result is true for $n = k + 1$. Thus for all positive integer we have

$$(r\cos\theta + ir\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta).$$

Case-2 If n is a negative integer.

Since n is a negative integer, therefore $n = -k$, where k being a positive integer.

$$\begin{aligned}
(r\cos\theta + ir\sin\theta)^n =& (r\cos\theta + ir\sin\theta)^{-k}\\
=& \frac{1}{(r\cos\theta + ir\sin\theta)^k}\\
=& \frac{1}{r^k(\cos k\theta + i\sin k\theta)} \quad (\text{by using equation (3.12)})\\
=& \frac{\cos k\theta - i\sin k\theta}{r^k(\cos k\theta + i\sin k\theta)(\cos k\theta - i\sin k\theta)}(\text{after rationalizing })\\
=& \frac{\cos k\theta - i\sin k\theta}{r^k(\cos^2 k\theta - i^2\sin^2 k\theta)}\\
=& r^{-k}(\cos k\theta - i\sin k\theta)\\
=& r^{-k}(\cos(-k)\theta + i\sin(-k)\theta)\\
=& r^n(\cos n\theta + i\sin n\theta)
\end{aligned}$$

Thus, for all negative integer n we have

$$(r\cos\theta + ir\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta).$$

□

Remark. 1. $(\sin\theta + i\cos\theta) = i(\cos\theta - i\sin\theta)$

2. If $z = (\cos\theta + i\sin\theta)$, then $\frac{1}{z} = (\cos\theta - i\sin\theta)$

Proof.

$$\begin{aligned}\frac{1}{z} &= \frac{1}{(\cos\theta + i\sin\theta)}\\ &= \frac{1}{(\cos\theta + i\sin\theta)} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}\\ &= \frac{(\cos\theta - i\sin\theta)}{\cos^2\theta + \sin^2\theta}\\ &= (\cos\theta - i\sin\theta)\end{aligned}$$

□

3. $\dfrac{1}{(\cos\theta - i\sin\theta)} = (\cos\theta + i\sin\theta)$ (by rationalization)
4. $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

   Proof.

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-n} &= \cos(-n\theta) + i\sin(-n\theta)\\ &= \cos n\theta - i\sin n\theta\\ &\quad \because \cos(-\theta) = \cos\theta\\ &\quad \sin(-\theta) = -\sin\theta\end{aligned}$$

□

5. $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$
6. $(\cos\theta - i\sin\theta)^{n} = \cos n\theta - i\sin n\theta$
7. $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

   Proof.

$$\begin{aligned}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\end{aligned}$$

□

8. $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2) = \cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)$
9. $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2).....(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1+\theta_2......+\theta_n) + i\sin(\theta_1+\theta_2......+\theta_n)$

Example 3.7. Find the modulus and the argument of the complex number

$$z = \frac{(\cos 4\theta + i\sin 4\theta)^7}{(\cos\theta + i\sin\theta)^5}$$

Solution. Given

$$z = \frac{(\cos 4\theta + i\sin 4\theta)^7}{(\cos\theta + i\sin\theta)^5} \tag{3.13}$$

We know, by De-Moivre's theorem,

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

Using this in equation (3.13)

$$\begin{aligned}
z =& \frac{\cos(4\times 7)\theta + i\sin(4\times 7)\theta}{(\cos 5\theta + i\sin 5\theta)} \\
=& \frac{\cos 28\theta + i\sin 28\theta}{(\cos 5\theta + i\sin 5\theta)} \\
=& (\cos 28\theta + i\sin 28\theta)(\cos 5\theta - i\sin 5\theta) \qquad \text{(using remark (iii))} \\
=& \cos(28-5)\theta + i\sin(28-5)\theta \qquad \text{(using remark (viii))} \\
=& (\cos 23\theta + i\sin 23\theta).
\end{aligned}$$

Thus,

$$z = (\cos 23\theta + i\sin 23\theta) \tag{3.14}$$

Comparing equation (3.14) by $z = x + iy$, we get $x = \cos 23\theta$, $y = \sin 23\theta$. So, modulus and arguments are:

$$\begin{aligned}
r =& \sqrt{x^2 + y^2} \\
=& \sqrt{\cos^2 23\theta + \sin^2 23\theta} \\
=& 1
\end{aligned}$$

$$\begin{aligned}
\phi =& \tan^{-1}\left(\frac{y}{x}\right) \\
=& \tan^{-1}\left(\frac{\sin 23\theta}{\cos 23\theta}\right) \\
=& \tan^{-1}(\tan 23\theta) \\
=& 23\theta
\end{aligned}$$

Also we know that polar form of complex number z is $z = r(\cos\phi + i\sin\phi)$. Comparing this with equation (3.14), we get the value of modulus is 1 and value of argument is 23θ.

Example 3.8. Express the following in the form of $x + iy$ and also find the modulus and argument of the complex number

$$z = \frac{(1 + \cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^2}.$$

Solution. To apply De Moivre's theorem, we need to convert it in the form of $(\cos\theta + i\sin\theta)^n$. For this, we know $1+\cos\theta = 2\cos^2\frac{\theta}{2}$, and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\begin{aligned}
z =& \frac{(1+\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^2} \\
=& \frac{\left(2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^4}{i^2(\cos\theta - i\sin\theta)^2} \\
=& \frac{\left(2\cos\frac{\theta}{2}\right)^4\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)^4}{-(\cos 2\theta - i\sin 2\theta)} \\
=& -\frac{\left(2\cos\frac{\theta}{2}\right)^4\left(\cos(\frac{4\theta}{2}) + i\sin(\frac{4\theta}{2})\right)}{(\cos 2\theta - i\sin 2\theta)} \\
=& -\left(2\cos\frac{\theta}{2}\right)^4(\cos(2\theta) + i\sin(2\theta))(\cos 2\theta + i\sin 2\theta) \\
=& \left(-16\cos^4\frac{\theta}{2}\right)(\cos(4\theta) + i\sin(4\theta)).
\end{aligned}$$

Thus,

$$z = \left(-16\cos^4\frac{\theta}{2}\right)(\cos(4\theta) + i\sin(4\theta)). \tag{3.15}$$

We know that polar form of complex number z is $z = r(\cos\theta + i\sin\theta)$. Comparing this with equation (3.15), we get the value of modulus is $\left(-16\cos^4\frac{\theta}{2}\right)$ and value of argument is 4θ.

Example 3.9. If $y_p = \cos\left(\frac{\pi}{2^p}\right) + i\sin\left(\frac{\pi}{2^p}\right)$. Prove that $y_1.y_2.y_3.........\infty = -1$

Solution.

$$\begin{aligned}
y_1 =& \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \\
y_2 =& \cos\left(\frac{\pi}{2^2}\right) + i\sin\left(\frac{\pi}{2^2}\right) \\
y_3 =& \cos\left(\frac{\pi}{2^3}\right) + i\sin\left(\frac{\pi}{2^3}\right) \\
& \vdots
\end{aligned} \tag{3.16}$$

$$\therefore y_1.y_2.y_3.........\infty = \left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right).\left(\cos\left(\frac{\pi}{2^2}\right) + i\sin\left(\frac{\pi}{2^2}\right)\right). \tag{3.17}$$
$$\left(\cos\left(\frac{\pi}{2^3}\right) + i\sin\left(\frac{\pi}{2^3}\right)\right)...\infty$$

$$\begin{aligned}
=& \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3}...\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3}...\right) \\
=& \cos(S) + i\sin(S)
\end{aligned} \tag{3.18}$$

where $S = \frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3}$.... Since this series form a infinite G.P. whose first term is $a = \frac{\pi}{2}$ and common ratio is $r = \frac{\frac{\pi}{2^2}}{\frac{\pi}{2}} = \frac{1}{2}$. Thus the sum of $S = \frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3}... = \frac{a}{1-r} = \frac{\frac{\pi}{2}}{1-\frac{1}{2}} = \pi$.

$$\begin{aligned} \text{Hence} \quad y_1.y_2.y_3.........\infty &= \cos(S) + i\sin(S) \\ &= \cos(\pi) + i\sin(\pi) \\ &= -1. \end{aligned} \tag{3.19}$$

Example 3.10. If $z = \cos\theta + i\sin\theta$, then prove that $z^p + \frac{1}{z^p} = 2\cos p\theta$ and $z^p - \frac{1}{z^p} = 2i\sin p\theta$.

Solution. Since

$$z = \cos\theta + i\sin\theta$$

$$z^p = \cos p\theta + i\sin p\theta \tag{3.20}$$

$$\frac{1}{z^p} = \cos p\theta - i\sin p\theta \tag{3.21}$$

Thus, adding equation (3.20) and (3.21), we get

$$z^p + \frac{1}{z^p} = 2\cos p\theta$$

By subtracting equation (3.20) and (3.21), we get

$$z^p - \frac{1}{z^p} = 2i\sin p\theta.$$

In-text Exercise 3.2. Solve the following

(a) If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$, prove that

$$\frac{a-b}{a+b} = i\tan\left(\frac{\alpha-\beta}{2}\right).$$

(b) Find the value of $(\sqrt{3}+i)^4$.

(c) Prove that $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos(\frac{n\pi}{4})$.

(d) If two roots of the equation $x^2 - 2x + 2 = 0$ are α and β, prove that

$$\alpha^n - \beta^n = 2^{\frac{n}{2}+1} i\sin\left(\frac{n\pi}{4}\right).$$

Theorem 3.2 (De Moivre's theorem for rational index)**.** *It states that if n is a rational/fractional number i.e.($n = \frac{p}{q}$, where p & q be integers and $q \neq 0$), then one of the values of* $(\cos\theta + i\sin\theta)^n$ *is* $(\cos n\theta + i\sin n\theta)$.

Proof. By De Moivre's theorem for integral index, we know

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \left(\cos\frac{q\,\theta}{q} + i\sin\frac{q\,\theta}{q}\right).$$
$$= (\cos\theta + i\sin\theta)$$

Thus qth power of $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)$ is $(\cos\theta + i\sin\theta)$.

i.e. $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)$ is one of the qth roots of $(\cos\theta + i\sin\theta)$.

i.e. $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)$ is one of the value of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$.

Raising both side the pth power, we have

$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p$ is one of the value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$.
or $\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$ is one of the value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$.
Hence one of the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$, where n is rational number. □

Remark. (i) In case of integer power of a complex number, there is one and only one value of $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$. But in case of rational power $(n = \frac{p}{q})$, there are q values of $(\cos\theta + i\sin\theta)^n$ one of which is $(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta)$ or $(\cos n\theta + i\sin n\theta)$.

(i) This theorem is also true for irrational numbers. In that case there are infinite number of values.

Theorem 3.3. *Show that all the values of* $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ *are given by* $(\cos(2r\pi + \theta)\frac{p}{q} + i\sin(2r\pi + \theta)\frac{p}{q})$, *where* $r = 0, 1, 2...q - 1$.

or

Show that $(\cos p\theta + i\sin p\theta)^{\frac{1}{q}}$ *has total* q *different values.*

Proof. Proof is omitted here. □

3.5 Applications of De Moivre's Theorem

The primary use of De Moivre's Theorem is to obtain the roots of a complex number, solution of some equations and relationship between some the powers and angle of trigonometric expressions etc.

3.5.1 To find roots of complex number

Example 3.11. Write down all the values of $(1+i)^{\frac{1}{4}}$.

Solution. Let the complex number

$$(1+i) = r(\cos\theta + i\sin\theta)$$

then on comparing real and imaginary parts, we have

$$r\cos\theta = 1 \tag{3.22}$$
$$r\sin\theta = 1 \tag{3.23}$$

Squaring on both sides of the equations (3.22) and (3.23) and adding, we get

$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta &= 1+1 \\ r^2 &= 2 \\ r &= \sqrt{2} \end{aligned}$$

Dividing equation (3.23) by equation (3.22), we get

$$\begin{aligned} \tan\theta &= 1 \\ \theta &= \tan^{-1}(1) = \frac{\pi}{4} \end{aligned}$$

Thus, $(1+i) = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$. Now using De Moivre's theorem for rational index

$$\begin{aligned} (1+i)^{\frac{1}{4}} &= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{\frac{1}{4}} \\ &= (\sqrt{2})^{\frac{1}{4}}\left(\cos\left(2k\pi + \frac{\pi}{4}\right).\frac{1}{4} + i\sin\left(2k\pi + \frac{\pi}{4}\right).\frac{1}{4}\right), \text{where k = 0, 1, 2, 3.} \end{aligned}$$

Thus, the four values of $(1+i)^{\frac{1}{4}}$ are

$$\begin{aligned} &\text{for k=0,} \quad (\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{16}\right)\right] \\ &\text{for k=1,} \quad (\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{9\pi}{16}\right) + i\sin\left(\frac{9\pi}{16}\right)\right] \\ &\text{for k=2,} \quad (\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{17\pi}{16}\right) + i\sin\left(\frac{17\pi}{16}\right)\right] \\ &\text{for k=3,} \quad (\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{25\pi}{16}\right) + i\sin\left(\frac{25\pi}{16}\right)\right] \end{aligned}$$

Note:- It was observed that the 3^{rd} value $(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{17\pi}{16}\right)+i\sin\left(\frac{17\pi}{16}\right)\right]$ can be written as

$$(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{17\pi}{16}\right)+i\sin\left(\frac{17\pi}{16}\right)\right]=(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\pi+\frac{\pi}{16}\right)+i\sin\left(\pi\frac{\pi}{16}\right)\right]$$

$$\text{Since}\qquad \cos(\pi+\theta)=-\cos\theta$$
$$\sin(\pi+\theta)=-\sin\theta$$

$$\therefore(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{17\pi}{16}\right)+i\sin\left(\frac{17\pi}{16}\right)\right]=(\sqrt{2})^{\frac{1}{4}}\left[-\cos\left(\frac{\pi}{16}\right)-i\sin\left(\frac{\pi}{16}\right)\right] \tag{3.24}$$
$$=-(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{\pi}{16}\right)+i\sin\left(\frac{\pi}{16}\right)\right]$$

which is negative of the first value.

Similarly $(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{25\pi}{16}\right)+i\sin\left(\frac{25\pi}{16}\right)\right]$ can be written as

$$(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{25\pi}{16}\right)+i\sin\left(\frac{25\pi}{16}\right)\right]=-(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{9\pi}{16}\right)+i\sin\left(\frac{9\pi}{16}\right)\right]$$

Thus the four values of $(1+i)^{\frac{1}{4}}$ can be written as

$$\pm(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{\pi}{16}\right)+i\sin\left(\frac{\pi}{16}\right)\right]$$
$$\pm(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{9\pi}{16}\right)+i\sin\left(\frac{9\pi}{16}\right)\right]$$

or simply, $\pm(\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{p\pi}{16}\right)+i\sin\left(\frac{p\pi}{16}\right)\right]$, where $p=1$ and 9.

Example 3.12. Find all the values of $(\frac{1}{2}+i\frac{\sqrt{3}}{2})^{\frac{3}{4}}$.

Solution. Let the complex number

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}=r(\cos\theta+i\sin\theta).$$

On comparing real and imaginary parts, we have

$$r\cos\theta=\frac{1}{2} \tag{3.25}$$
$$r\sin\theta=\frac{\sqrt{3}}{2} \tag{3.26}$$

Squaring on both sides of the equations (3.25) and (3.26) and adding, we get

$$r^2\cos^2\theta+r^2\sin^2\theta=\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$
$$r^2=1$$
$$r=+1$$

(as r is the length of the vector so it can not be negative). Also dividing equation (3.26) by equation (3.25), we get

$$\frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{3}/2}{1/2}$$
$$\tan\theta = \sqrt{3}$$
$$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

Therefore, the complex number

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

Now,

$$\begin{aligned}
\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} &= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} \\
&= \left(\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{3}\right)^{\frac{1}{4}} \\
&= \left(\cos 3\left(\frac{\pi}{3}\right) + i\sin 3\left(\frac{\pi}{3}\right)\right)^{\frac{1}{4}} \qquad \text{(using De Moivre's theorem for integer index)} \\
&= (\cos\pi + i\sin\pi)^{\frac{1}{4}}.
\end{aligned}$$

Using De Moivre's theorem for rational index, we get

$$(\cos\pi + i\sin\pi)^{\frac{1}{4}} = \left(\cos\left(2k\pi + \pi\right).\frac{1}{4} + i\sin\left(2k\pi + \pi\right).\frac{1}{4}\right) \qquad \text{where k = 0, 1, 2, 3.}$$

Thus the obtained four values are

$$\begin{aligned}
\text{for k=0,} \quad P_1 &= \left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \\
\text{for k=1,} \quad P_2 &= \left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] \\
\text{for k=2,} \quad P_3 &= \left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] \\
\text{for k=3,} \quad P_4 &= \left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]
\end{aligned}$$

The product of above four values is

$$\begin{aligned}
P_1.P_2.P_3.P_4 &= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) \\
&= \cos(4\pi) + i\sin(4\pi) \\
&= 1
\end{aligned}$$

3.5.2 Expansions of $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$

By De Moivre's theorem, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{3.27}$$

Also,

$$(\cos\theta + i\sin\theta)^n = {}^nC_0\cos^n\theta + {}^nC_1\cos^{n-1}\theta(i\sin\theta)^1 + {}^nC_2\cos^{n-2}\theta(i\sin\theta)^2 + \\ ... + {}^nC_n\cos^{n-n}\theta(i\sin\theta)^n$$

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + ... + (i)^n\sin n\theta \tag{3.28}$$

From equation (3.27) and (3.28), we get

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + ... + (i)^n\sin n\theta \tag{3.29}$$

On comparing real and imaginary part, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + ... \tag{3.30}$$
$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + ... \tag{3.31}$$

This is the required relation between $\cos n\theta$ and $\sin n\theta$ and powers of $cos\theta$ and $\sin\theta$.

Example 3.13. Write down the expansion of $\cos 5\theta$, $\sin 5\theta$ and $\tan 5\theta$ in terms of powers of $cos\theta$, $\sin\theta$. and $\tan\theta$.

Solution. Putting $n = 5$ in equation (3.29), we get

$$\cos 5\theta + i\sin 5\theta = \cos^5\theta + i\,{}^5C_1\cos^4\theta\sin\theta - {}^5C_2\cos^3\theta\sin^2\theta - i\,{}^5C_3\cos^2\theta\sin^3\theta \\ + {}^5C_4\cos^1\theta\sin^4\theta + i\,{}^5C_5\sin^5\theta \tag{3.32}$$

On comparing real and imaginary part, we get

$$\cos 5\theta = \cos^5\theta - {}^5C_2\cos^3\theta\sin^2\theta + + {}^5C_4\cos^1\theta\sin^4\theta$$
$$\sin 5\theta = {}^5C_1\cos^4\theta\sin\theta - {}^5C_3\cos^2\theta\sin^3\theta + {}^5C_5\sin^5\theta$$

On solving, we get

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \tag{3.33}$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \tag{3.34}$$

Dividing equation (3.34) by equation (3.33), we get

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \tag{3.35}$$

Dividing the numerator and denominator of equation (3.35) by $\cos^5\theta$, we get

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} \tag{3.36}$$

3.5.3 Expansions of $\cos^n\theta$ and $\sin^n\theta$ in terms of sines and cosines of multiples of θ

Let

$$z = \cos\theta + i\sin\theta \tag{3.37}$$

then

$$\frac{1}{z} = \cos\theta - i\sin\theta \tag{3.38}$$

Also

$$z^n = \cos n\theta + i\sin n\theta \tag{3.39}$$

then

$$\frac{1}{z^n} = \cos n\theta - i\sin n\theta. \tag{3.40}$$

Adding and subtracting equation (3.39) and (3.40), we get

$$z^n + \frac{1}{z^n} = 2\cos n\theta \tag{3.41}$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta \tag{3.42}$$

Also from equation (3.37) and (3.38), we have

$$\left(z + \frac{1}{z}\right)^n = (2\cos\theta)^n \tag{3.43}$$

$$\left(z - \frac{1}{z}\right)^n = (2i\sin\theta)^n \tag{3.44}$$

Expanding the left hand side of equation (3.43) by binomial expansion and using the equation (3.41), we get the required relation between $\cos^n\theta$ and cosines of multiples of θ. Similarly expanding the left hand side of equation (3.44) and using the equation (3.42), we get the required relation between $\sin^n\theta$ and sines of multiples of θ

Example 3.14. Prove that $\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = 32\cos^6\theta$.

Solution. Expanding the left hand side of equation (3.43) by binomial expansion for $(n = 6)$

$$\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6$$

$$z^6 + {}^6C_1 z^5\left(\frac{1}{z}\right) + {}^6C_2 z^4\left(\frac{1}{z}\right)^2 + {}^6C_3 z^3\left(\frac{1}{z}\right)^3 + {}^6C_4 z^2\left(\frac{1}{z}\right)^4 + {}^6C_5 z\left(\frac{1}{z}\right)^5 + {}^6C_6\left(\frac{1}{z}\right)^6 = 2^6\cos^6\theta$$

$$z^6 + 6z^4 + 15z^2 + 20 + 15\left(\frac{1}{z}\right)^2 + 6\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^6 = 2^6\cos^6\theta$$

$$\left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20 = 2^6\cos^6\theta$$

Using equation (3.41), we get

$$\begin{aligned} 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20 &= 2^6\cos^6\theta \\ \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 &= 32\cos^6\theta \end{aligned}$$

is the required relation between $\cos^n\theta$ and cosines of multiples of θ.

In-text Exercise 3.3. Solve the following

(a) Find all the values of $(i)^{\frac{1}{8}}$.

(b) Find all the values of $(\sqrt{3}-i)^{\frac{2}{5}}$.

(c) Expand $\cos 7\theta$, $\sin 7\theta$ in terms of powers of $cos\theta$ and $\sin\theta$.

(d) Prove that $-32\sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10$.

3.6 n^{th} Roots of Unity

The solution of the equation $z^n = 1$, for positive values of integer , called the n^{th} roots of unity or a root of unity also known as De Moivre's number is a complex number that gives the value 1 when raised to some positive integer power n. Mathematically, the solution of $z^n = 1$ or $z = (1)^{\frac{1}{n}}$ is called the n^{th} roots of unity.

Example 3.15. Find the n^{th} roots of unity.

Solution. Given $z^n = 1$

$$\begin{aligned} z =& (1)^{\frac{1}{n}} \\ =& (\cos 0 + i\sin 0)^{\frac{1}{n}} \\ =& \cos\left((2r\pi+0).\frac{1}{n}\right) + i\sin\left((2r\pi+0).\frac{1}{n}\right), \qquad \text{(By using theorem 3.3)} \\ & \text{where } r = 0,1,2,......n-1 \\ =& \cos\left(\frac{2r\pi}{n}\right) + i\sin\left(\frac{2r\pi}{n}\right), \qquad \text{where } r = 0,1,2,......n-1. \end{aligned}$$

Thus, n^{th} roots of unity are
$\cos 0 + i\sin 0,\ \cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right),\ \cos\left(\frac{4\pi}{n}\right)+i\sin\left(\frac{4\pi}{n}\right), ..., \cos\left(\frac{2(n-1)\pi}{n}\right)+i\sin\left(\frac{2(n-1)\pi}{n}\right)$

Let $\alpha = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ then $\alpha^2 = \cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right)$. Therefore, nth roots of unity are found as $1, \alpha, \alpha^2 ..., \alpha^{n-1}$.

Example 3.16. Show that the n^{th} roots of unity form a series in G.P.

Solution. Since n^{th} roots of unity are $1, \alpha, \alpha^2 ... \alpha^{n-1}$, where $\alpha = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$. The given series form a G.P. whose first term is 1 and common ratio is α. The sum of the series is found as

$$\begin{aligned}1+\alpha+\alpha^2...\alpha^{n-1} &= \frac{1.(1-\alpha^n)}{1-\alpha} \\ &= \frac{\left(1-\left[\cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right)\right]^n\right)}{1-\left[\cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right)\right]} \\ &= \frac{\left(1-\left[\cos\left(\frac{2n\pi}{n}\right)+i\sin\left(\frac{2n\pi}{n}\right)\right]\right)}{1-\left[\cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right)\right]} \\ &= \frac{(1-1)}{1-\left[\cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right)\right]} \\ &= 0.\end{aligned}$$

$$\begin{aligned}\alpha^n &= \left[\cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right)\right]^n \\ &= \left[\cos\left(\frac{2n\pi}{n}\right)+i\sin\left(\frac{2n\pi}{n}\right)\right] \\ &= \cos(2\pi)+i\sin(2\pi) \\ &= 1\end{aligned}$$

Sum of the n^{th} roots of unity can also be found by using relation between roots and coefficients as $1, \alpha, \alpha^2 ... \alpha^{n-1}$ are the roots of $z^n - 1 = 0$. Thus

$$\text{Sum of the roots} = 1+\alpha+\alpha^2...+\alpha^{n-1} = -\frac{\text{coefficient of} z^{n-1}}{\text{coefficient of} z^n} = 0.$$

$$\text{product of the roots} = 1.\alpha.\alpha^2.\alpha^3+...+\alpha^{n-1} = (-1)^n\frac{\text{constant term}}{\text{coefficient of} z^n} = (-1)^{n-1}.$$

Thus the sum of all the n roots of unity is 0 and product of all the roots of unity is $(-1)^{n-1}$, also $\alpha^n = 1$.

Example 3.17. Solve the equation $z^4 + 1 = 0$

Solution. Given $z^4 + 1 = 0$

$$\begin{aligned}z =&(-1)^{\frac{1}{4}} \\ =&(\cos\pi + i\sin\pi)^{\frac{1}{4}} \\ =&\cos\left((2r\pi+\pi).\frac{1}{4}\right)+i\sin\left((2r\pi+\pi).\frac{1}{4}\right), \qquad \text{(By using theorem 3.3)} \\ &\text{where } r = 0,1,2,3. \\ =&\cos\left(\frac{(2r+1)\pi}{4}\right)+i\sin\left(\frac{(2r+1)\pi}{4}\right), \qquad \text{where } r = 0,1,2,3.\end{aligned}$$

Thus the four roots of $z^4 + 1 = 0$, are

$$\begin{aligned}
&\text{when r=0,} && \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\
&\text{when r=1,} && \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\
&\text{when r=2,} && \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \\
&\text{when r=3,} && \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}
\end{aligned}$$

i.e. $\pm\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$ are the four roots of the given equation.

Remark.

1. The n^{th} roots of unity lie on the circumference of the circle, whose radius is equal to 1 and center is the origin $(0, 0)$.
2. The sum of all the roots of unity is 0 and product of all the roots of unity is $(-1)^{n-1}$, also $\alpha^n = 1$.
3. The n^{th} roots of unity $1, \alpha, \alpha^2 ... \alpha^{n-1}$ are in geometric progression called G.P.

In-text Exercise 3.4. Solve the following

(a) Find the cube roots of unity. Also find the sum and product of the roots.

(b) Solve the equation $z^6 + z^3 + 1 = 0$.

(c) Solve the equation $x^7 - x^4 + x^3 - 1 = 0$.

(d) Find the fifth roots of unity and verify that the obtained roots form a geometric progression.

3.7 Summary

In the end of the chapter, we know

1. De Moivre's theorem for integer and fraction index.
2. De Moivre's theorem gives a formula for computing powers of complex numbers.
3. This theorem is also helpful for discovering correlation between the functions of numerous angles that are calculated using trigonometry.
4. Using this theorem we can find the n^{th} roots of unity.

3.8 Self-Assessment Exercises

1. Write the trigonometric representation of $1 + \cos\alpha + i\sin\alpha$.

2. If $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$, where n is a positive integer. Then prove that

   (a) ${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + ...x^n = 2^{(\frac{n}{2})}\cos(\frac{n\pi}{4})$

   (b) ${}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + ...x^n = 2^{(\frac{n}{2})}\sin(\frac{n\pi}{4})$

3. If $\sin\alpha + \sin\beta + \sin\gamma = 0$ and $\cos\alpha + \cos\beta + \cos\gamma = 0$. Prove that

   (a) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

   (b) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

   (c) $\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0$

   (d) $\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0$

4. Express $\frac{(\cos\theta + i\sin\theta)^7}{(\cos 2\theta + i\sin 2\theta)^2}$ in the form of $a + ib$.

5. Write down all the values of $(1 + i\sqrt{3})^{\frac{3}{5}}$.

6. Expand $\sin^7\theta\cos^2\theta$ in a series of sines of multiple of θ.

7. Prove that $256\sin^5\theta\cos^4\theta = \sin 9\theta - \sin 7\theta - 4\sin 5\theta + 4\sin 3\theta + 6\sin\theta$.

8. Express $\cos^4\theta$ in terms of cosines of multiples of θ.

9. Solve the expression $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0$.

10. Solve the equation $z^9 + z^5 + z^4 + 1 = 0$.

11. Find the fifth roots of (-32).

3.9 Solutions to In-text Exercises

Exercise 3.1

(a) Roots obtained by using quadratic formula are $x = 3 \pm i$.

(b) Complex number $5 + 2i$ has modulus $r = \sqrt{5^2 + 2^2} = \sqrt{29}$ and argument as $\theta = \tan^{-1}\left(\frac{2}{5}\right)$. Thus the polar form of the given complex number is

$$5 + 2i = \sqrt{29}\left(\cos\left(\tan^{-1}\left(\frac{2}{5}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{2}{5}\right)\right)\right)$$
$$5 + 2i \approx 5.3(\cos(0.38) + i\sin(0.38)).$$

(c)

$$z = \frac{2+3i}{1+i} \times \frac{1-i}{1-i} = \frac{5+i}{2}.$$

(d) Let $z = \cos\theta + i\sin\theta$, then $|z| = \sqrt{\cos\theta^2 + \sin\theta^2} = 1$.

Exercise 3.2

(a) Since

$$\begin{aligned}
a+b &= (\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta) \\
a+b &= 2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right] \\
a-b &= (\cos\alpha - \cos\beta) + i(\sin\alpha - \sin\beta) \\
a-b &= 2i\sin\left(\frac{\alpha-\beta}{2}\right)\left[\cos(\frac{\alpha+\beta}{2}) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]
\end{aligned}$$

Hence $\frac{a-b}{a+b} = i\tan\left(\frac{\alpha-\beta}{2}\right)$.

(b) Since

$$\begin{aligned}
(\sqrt{3}+i)^4 &= \left(2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]\right)^4 \\
&= 16\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] \\
&= -8 + i8\sqrt{3}
\end{aligned}$$

(c) Since

$$\begin{aligned}
1+i &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\
1-i &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right)
\end{aligned}$$

Therefore

$$(1+i)^n = 2^{\frac{n}{2}}\left(\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right) \tag{3.45}$$

$$(1-i)^n = 2^{\frac{n}{2}}\left(\cos\left(\frac{n\pi}{4}\right) - i\sin\left(\frac{n\pi}{4}\right)\right) \tag{3.46}$$

After adding both the equation (3.45) and (3.46), we get the required result.

(d) In the above question by taking $\alpha = 1+i$ and $\beta = 1-i$ and subtracting equation (3.46) from (3.45), we get the required result.

Exercise 3.3

(a) $(i)^{\frac{1}{8}} = (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^{\frac{1}{8}}$. Now using De Moivre's theorem for rational index

$$\begin{aligned}(i)^{\frac{1}{8}} &= \left(\cos\left(2k\pi + \frac{\pi}{2}\right).\frac{1}{8} + i\sin\left(2k\pi + \frac{\pi}{2}\right).\frac{1}{8}\right) \text{ where k = 0, 1, 2, 3, 4, 5, 6, 7.}\\ &= \cos\left((4k+1)\frac{\pi}{16}\right) + i\sin\left((4k+1)\frac{\pi}{16}\right) \text{ where k = 0, 1, 2, 3, 4, 5, 6, 7.}\end{aligned}$$

(b) $(\sqrt{3} - i)^{\frac{2}{5}} = (2)^{\frac{2}{5}}(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})^{\frac{2}{5}}$. Now using De Moivre's theorem for rational index

$$\begin{aligned}(\sqrt{3} - i)^{\frac{2}{5}} &= (2)^{\frac{2}{5}}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^{\frac{1}{5}}\\ &= (2)^{\frac{2}{5}}\left(\cos\left(2k\pi - \frac{\pi}{3}\right).\frac{1}{5} + i\sin\left(2k\pi - \frac{\pi}{3}\right).\frac{1}{5}\right) \text{ where k = 0, 1, 2, 3, 4.}\end{aligned}$$

(c) Putting $n = 7$ in equation (3.29), we get

$$\begin{aligned}\cos 7\theta + i\sin 7\theta = \cos^7\theta + i\,{}^7C_1\cos^6\theta\sin\theta - {}^7C_2\cos^5\theta\sin^2\theta - i\,{}^7C_3\cos^4\theta\sin^3\theta\\ + {}^7C_4\cos^3\theta\sin^4\theta + i\ {}^7C_5\cos^2\theta\sin^5\theta - {}^7C_6\cos^1\theta\sin^6\theta - i\ {}^7C_7\sin^7\theta.\end{aligned} \tag{3.47}$$

On comparing real and imaginary part, we get the required result.

(d) Expanding the left hand side of equation (3.44) by binomial expansion for $(n = 6)$

$$\begin{aligned}\left(z - \frac{1}{z}\right)^6 &= (2i\sin\theta)^6\\ z^6 - {}^6C_1z^5\left(\frac{1}{z}\right) + {}^6C_2z^4\left(\frac{1}{z}\right)^2 - {}^6C_3z^3\left(\frac{1}{z}\right)^3 + {}^6C_4z^2\left(\frac{1}{z}\right)^4 &\\ - {}^6C_5z\left(\frac{1}{z}\right)^5 + {}^6C_6\left(\frac{1}{z}\right)^6 &= -2^6\sin^6\theta\\ z^6 - 6z^4 + 15z^2 - 20 + 15\left(\frac{1}{z}\right)^2 - 6\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^6 &= -2^6\sin^6\theta\\ \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20 &= -2^6\sin^6\theta\end{aligned}$$

Using equation (3.41), we get

$$\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10 = -32\sin^6\theta.$$

Exercise 3.4

(a) To find the cube root of unity we have to solve $z^3 - 1 = 0$ or $z = (1)^{\frac{1}{3}}$.

$$\begin{aligned} z =& (1)^{\frac{1}{3}} \\ =& (\cos 0 + i \sin 0)^{\frac{1}{3}} \\ =& \cos\left((2r\pi + 0).\frac{1}{3}\right) + i \sin\left((2r\pi + 0).\frac{1}{3}\right), \qquad \text{(By using theorem 3.3)} \\ & \text{where } r = 0, 1, 2 \\ =& \cos\left(\frac{2r\pi}{3}\right) + i \sin\left(\frac{2r\pi}{3}\right), \qquad \text{where } r = 0, 1, 2. \end{aligned}$$

Thus the three roots of unity are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

(b) Since the given equation is quadratic in z^3, therefore we get

$$\begin{aligned} z^3 &= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \\ z^3 &= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = r(\cos\theta \pm i \sin\theta) \\ z &= \left[\cos(2r\pi + \frac{2\pi}{3}) \pm i \sin(2r\pi + \frac{2\pi}{3})\right]^{\frac{1}{3}} \qquad \text{where } r = 0, 1, 2. \end{aligned}$$

(c) Since the given equation is $x^7 - x^4 + x^3 - 1 = 0$.

$$\begin{aligned} x^7 - x^4 + x^3 - 1 &= 0 \\ (x^4 + 1)(x^3 - 1) &= 0 \\ (x^4 + 1) = 0, (x^3 - 1) &= 0. \end{aligned}$$

By example 3.17 and cube root of unity, the seven roots are given as

$$1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \pm\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}.$$

(d) Fifth roots of unity can be obtained same as the cube roots of unity.

Suggested Readings

1. Burnside, W.S., & Panton, A.W. (1979). The Theory of Equations. Vol. 1. Eleventh Edition, (Fourth Indian Reprint. S. Chand & Co. New Delhi), Dover Publications, Inc.

2. Dickson, Leonard Eugene (2009). First Course in the Theory of Equations. John Wiley & Sons, Inc. The Project Gutenberg eBook (http://www.gutenberg.org/ebooks/29785).

3. Prasad, Chandrika (2017). Text Book of Algebra and Theory of Equations. Pothishala Pvt Ltd.

Lesson - 4

Cubic and Biquadratic Equations

Structure

4.1 Learning Objectives

After studying this chapter, student should be able to

- better understanding of transformation of equations.
- understand the relationship between roots and the coefficients of the equations.

- apply Cardano's method for calculating the roots of cubic equations.
- apply Descartes' method for calculating the roots of biquadratic equations.

4.2 Introduction

Babylonians and other various mathematicians by 2000 B.C.E. devised methods for finding solutions of cubic equations. During the early and middle 16th centuries mathematicians discovered formulas for the roots of cubic and biquadratic (fourth degree) polynomials in terms of the coefficients of the polynomial.

Scipione dal Ferro (1465-1526) is credited with solving cubic equations. The problem was to find the roots by adding, subtracting, multiplying, dividing and taking roots of expressions in the coefficients. The original formula was discovered in one basic case but not published by Ferro. It was rediscovered independently and extended to other cases by Nicolo Fontana, who is better known as Tartaglia and G. Cardano's without permission from Tartaglia by G. Cardano Ain his book, "Ars Magna" which was devoted to algebra which is given as:

4.3 Algebraic solution of the cubic equation

4.3.1 Cardan's Method

An equation of the form

$$px^3 + qx^2 + rx + s = 0$$

is called the ***cubic equation*** i.e., an equation of $3rd$ degree with real coefficients.

Example 4.1. 1. $x^3 + x^2 + 1 = 0$ is a cubic equation with real coefficients.

2. $x^3 = 1$ is an equation of $3rd$ degree.
3. $6x^5 + x^2 = 9$ is not a cubic equation.

Let the general cubic equation be given by

$$px^3 + qx^2 + rx + s = 0 \tag{4.1}$$

Eliminating the square term with the help of the substitution $y = x + q/3p$, we obtain

$$\begin{aligned}
&(y - \frac{q}{3p})^3 + \frac{q}{p}(y - \frac{q}{3p})^2 + \frac{r}{p}(y - \frac{q}{3p}) + \frac{s}{p} = 0 \\
&\Rightarrow y^3 - qy^2 + \frac{q^2}{3p}y - \frac{q^3}{27p^2} + qy^2 - \frac{2q^2}{3p}y + \frac{q^3}{9p^2} + ry - \frac{qr}{3p} + \frac{s}{p} = 0 \\
&\Rightarrow py^3 + (r - \frac{q^2}{3p^2})y + (s + \frac{2q^3}{27p^3} - \frac{qr}{3p}) = 0 \\
&\Rightarrow y^3 + \frac{3pr - q^2}{3p^2}y + \frac{27p^2s + 2q^3 - 9pqr}{27p^3} = 0. \qquad (4.2)
\end{aligned}$$

For example, consider the equation $2x^3 - 30x^2 + 15x + 30 = 0$. Then by substituting $y = x - (-30)/(3 \times 2) = x + 5$, we obtain

$$
\begin{aligned}
&2(y^3 + 15y^2 + 75y + 125) - 30(y^2 + 10y + 25) + 15(y + 5) + 30 = 0\\
&2y^3 + (150 - 300 + 15)y + (250 - 750 + 75 + 30) = 0\\
&2y^3 - 135y - 395 = 0.
\end{aligned}
$$

Consider the equation (1.2). It can be written as

$$z^3 + 3Az - 2B = 0$$

where

$$
\begin{aligned}
A &= (r - \frac{q^2}{3p})\\
B &= (s + \frac{2q^3}{27p^2} - \frac{qr}{3p}).
\end{aligned}
$$

Now, let $z = u + v$ and $A = -uv$, we obtain

$$
\begin{aligned}
&(u + v)^3 + 3(-uv)(u + v) - 2B = 0\\
&\Rightarrow (u^3 + 3u^2v + 3uv^2 + v^3) - 3u(u + v) - 3v(u + v) - 2B = 0\\
&\Rightarrow u^3 + v^3 - 2B = 0\\
&\Rightarrow u^3 - \frac{A^3}{u^3} - 2B = 0\\
&\Rightarrow u^6 - A^3 - 2u^3 = 0\\
&\Rightarrow u^6 - 2Bu^3 - A^3 = 0.
\end{aligned}
$$

The above equation is quadratic in u^3, which has the solutions given by
$u^3 = B \pm \sqrt{B^3 + A^2}$.

Since $uv = -A$, we obtain that $v^3 = -\frac{A^3}{u^3}$ and $u^3 = B + \sqrt{B^3 + A^2}$. This gives that $v^3 = B - \sqrt{B^3 + A^2}$.

Therefore, we obtain

$$
\begin{aligned}
u^3 &= B + \sqrt{B^3 + A^2},\\
v^3 &= B - \sqrt{B^3 + A^2}\\
\Rightarrow u &= (B + \sqrt{B^3 + A^2})^(1/3) = \alpha,\\
v &= (B - \sqrt{B^3 + A^2})^(1/3) = \beta.
\end{aligned}
$$

Three cube roots of u^3 and v^3 are given by
$\omega\alpha, \omega^2\alpha, \omega\beta, \omega\beta^2$.

Therefore, the three roots of the given equation (4.3.1) are

$$
\begin{aligned}
x_1 &= \alpha + \beta - \frac{q}{3p},\\
x_2 &= -\frac{\alpha + \beta}{2} - \frac{q}{3p} + \frac{i\sqrt{3}}{2}(\alpha - \beta),\\
x_3 &= -\frac{\alpha + \beta}{2} - \frac{q}{3p} - \frac{i\sqrt{3}}{2}(\alpha - \beta).
\end{aligned}
$$

This solution is called ***Cardan's solution of the cubic***.

Example 4.2. Solve $2x^3 + 3x^2 + 6x + 4 = 0$.

Solution. We shall remove the second term in the given equation by completing the cube in the following way:

$$\begin{aligned} x^3 + \frac{3}{2}x^2 + 3x + 2 &= 0 \\ [(x + \frac{1}{2})^3 - \frac{3}{4}x - \frac{1}{8}] + 3x + 2 &= 0 \\ \Rightarrow (x + \frac{1}{2})^3 - \frac{3}{4}x + 3x + 2 - \frac{1}{8} &= 0 \\ \Rightarrow (x + \frac{1}{2})^3 + \frac{9}{4}x + 3x + \frac{15}{8} &= 0. \end{aligned}$$

Put $x + \frac{1}{2} = y$ in the above equation, we get

$$\begin{aligned} y^3 + \frac{9}{4}(y - \frac{1}{2}) + 3(y - \frac{1}{2})\frac{15}{8} &= 0 \\ \Rightarrow y^3 + \frac{9}{4}y - \frac{9}{8} + 3y - \frac{3}{2} + \frac{15}{8} &= 0 \\ \Rightarrow y^3 + \frac{9}{4}y + 3y - \frac{3}{2} + \frac{6}{8} &= 0 \\ \Rightarrow y^3 + \frac{21}{4}y - \frac{3}{4} &= 0. \end{aligned}$$

Let $y = m + n$ where $m, n \in \mathbb{C}$. Then

$$\begin{aligned} (m+n)^3 + \frac{21}{4}(m+n) - \frac{3}{4} &= 0 \\ m^3 + 3mn(m+n) + n^3 + \frac{21}{4}(m+n) - \frac{3}{4} &= 0 \\ \Rightarrow m^3 + n^3 + 3mn(m+n) + \frac{21}{4}(m+n) - \frac{3}{4} &= 0 \\ \Rightarrow m^3 + n^3 + 3(m+n)(mn + \frac{7}{4}) - \frac{3}{4} &= 0 \end{aligned}$$

This gives that $mn + \frac{7}{4} = 0$. This implies $mn = -\frac{7}{4}$. This further gives that $m^3 + n^3 = \frac{3}{4}$ and $m^3n^3 = \frac{27}{64}$.

We see that m^3 and n^3 are the roots of $t^2 + \frac{7}{4}t + \frac{27}{64} = 0$. Hence, by quadratic formula, we find that

$$\begin{aligned} m^3 &= \frac{-7 + 4\sqrt{\frac{49}{16} - \frac{27}{16}}}{8} = \alpha \\ n^3 &= \frac{-7 - 4\sqrt{\frac{49}{16} - \frac{27}{16}}}{8} = \beta. \end{aligned}$$

Hence, the roots of the equation are $\alpha + \beta, \alpha\omega + \beta\omega^2, \alpha\omega^2 + \beta\omega$.

In-text Exercise 4.1. Find the roots of the following cubic equations:

1. $x^3 - 6x - 9 = 0$.
2. $x^3 - 9x^2 + 14x + 24 = 0$.

4.3.2 Descartes' Solution of the Quartic Equation

The equation of general quartic is given by

$$x^4 + ax^3 + bx^2 + cx + d = 0. \tag{4.3}$$

Replacing $x = z - a/4$ in (4.3), we obtain

$$z^4 + pz^2 + qz + r = 0. \tag{4.4}$$

(4.3) is called the *reduced quartic equation*. We claim that (4.4) can be expressed as the product of two quadratic factors

$$(z^2 + kz + l)(z^2 - kz + m) = z^4 + (l + m - k^2)z^2 + k(l - m)z + lm$$

On equating the coefficients, we obtain

$$\begin{aligned} l + m - k^2 &= p \\ k(l - m) &= q \\ lm &= r. \end{aligned}$$

This gives that

$$\begin{aligned} l + m &= p + k^2, \\ l - m &= \frac{q}{k}. \end{aligned}$$

If $k \neq 0$, from first two equations, we obtain

$$\begin{aligned} 2l &= p + k^2 - \frac{q}{k} \\ 2m &= p + k^2 + \frac{q}{k} \end{aligned}$$

Putting the above values in $l \cdot m = r$, we have

$$k^6 + 2pk^4 + (p^2 - 4r)k^2 - q^2 = 0. \tag{4.5}$$

On putting $k^2 = t$, we get

$$t^3 + 2pt^2 + (p^2 - 4r)k - q^2 = 0.$$

This is a cubic equation with at least one positive real root. Thus (4.4) is equivalent to solving

$$\begin{aligned} &(z^2 + kz + l)(z^2 - kz + m) = 0 \\ &\Rightarrow (z^2 + kz + l) = 0 \text{ or } (z^2 - kz + m) = 0. \end{aligned}$$

The above quadratic equations will give 4 roots of (4.3).

Example 4.3. Solve $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$.

Solution. The above equation can be rewritten as

$$x^4 + 4(2)x^3 + 6\frac{3}{2}x^2 + 4(-2)x - 10 = 0.$$

We shall remove the x^3 term in the given equation using $x = z + h$ where $h = -\frac{b}{a}$ $= -\frac{2}{1} = -2$. So we get $x = z - 2$. Putting this value of x in the given equation, we get

$$\begin{aligned}
&(z-2)^4 + 8(z-2)^3 + 9(z-2)^2 - 8(z-2) - 10 = 0\\
&\Rightarrow z^4 - 8z^3 + 24z^2 - 32z + 16\\
&+ 8(z^3 - 48z^2 + 96z - 64) + 9(z^2 - 4z + 4) - 8(z-2) - 10 = 0\\
&\Rightarrow z^4 - 8z^3 + 24z^2 - 32z + 16\\
&+ 8z^3 - 48z^2 + 96z - 64 + 9z^2 - 36z + 36 - 8z + 16 - 10 = 0\\
&\Rightarrow z^4 - 15z^2 + 20z - 6 = 0
\end{aligned}$$

Then we write

$$z^4 - 15z^2 + 20z - 6 = (z^2 + kz + l)(z^2 - kz + m).$$

This gives that

$$\begin{aligned}
&l + m - k^2 = -15\\
&k(l - m) = 20\\
&lm = -6.
\end{aligned}$$

On eliminating l and m from the above equations, we get

$$\begin{aligned}
&(l+m)^2 - (l-m)^2 = 4lm\\
&(k-15)^2 - (\frac{20}{k})^2 = 4 \cdot -6\\
&k^6 - 30k^4 + 249k^2 - 400 = 0.
\end{aligned}$$

$k^2 = 16$ is the root of the above equation in k^2. Thus, $k = 4$. Using this value of k, we obtain $l = 3$ and $m = -2$. Therefore, the roots of the given equation are the roots of the equations $(x^2 + 4x + 3)(x^2 - 4x - 2) = 0$. This gives 4 roots as $1, 3$ and $-2 \pm i\sqrt{6}$. Thus the roots of the equation are $z - 2$ i.e. are $-1, 1, 2 \pm \sqrt{6}$.

Example 4.4. Solve $x^4 - 2x^2 + 8x - 3 = 0$.

Solution. Since there is no x^3 term in the given equation, we write the above equation as

$$x^4 - 2x^2 + 8x - 3 = (z^2 + kz + l)(z^2 - kz + m).$$

This gives that

$$l + m - k^2 = -2$$
$$k(l - m) = 8$$
$$lm = -3.$$

On eliminating l and m from the above equations, we get

$$(l + m)^2 - (l - m)^2 = 4lm$$
$$(k + 2)^2 - (\frac{8}{k})^2 = 4 \cdot 6$$
$$k^6 - 4k^4 + 16k^2 - 64 = 0.$$

$k^2 = 4$ is the root of the above equation in k^2. Thus, $k = 2$. Using this value of k, we obtain $l = 3$ and $m = -1$. Therefore, the roots of the given equation are the roots of the equations $x^2 + 2x - 3)(x^2 - 2x - 1) = 0$. This gives 4 roots as $-1 \pm \sqrt{2}$ and $-1 \pm i\sqrt{2}$.

In-text Exercise 4.2. Find the roots of the following quartic equations:

1. $x^4 - 2x^2 + 8x - 3 = 0$.
2. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$.

4.4 Transformation of Equations

Sometimes without knowing the roots of an equation in terms of its coefficients, we can transform one symmetric equation into another in which roots of the new equation has some relation with the roots of the previous equation.

4.4.1 To form an equation whose roots are the negatives of the roots of a given equation

To form an equation whose roots are the negatives of the roots of a given equation of degree n, multiply the coefficients of $x^n, x^{n-1}, \cdots$ by $1, -1, 1, -1, \cdots$.

Let $p(x) = a_0x + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of $p(x)$ then

$$p(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n). \tag{4.6}$$

Put $x = -y$, we have

$$a_0 - a_1y + a_2y^2 + \cdots + (-1)na_ny^n = (y + \alpha_1)(y + \alpha_2) \cdots (y + \alpha_n).$$

Thus the roots are $-\alpha_1, +\alpha_2, \cdots, -\alpha_n$. Therefore, the required equation is

$$a_0 - a_1x + a_2x^2 + \cdots + (-1)^n a_nx = 0$$

Note. To form an equation whose roots are the negative of the roots of the given equation, change the signs of every alternate term of the given equation written in decreasing powers of x.

Example 4.5. Find the equation whose roots are the negatives of the roots of the equation $x^3 - 3x^2 + 5x - 4 = 0$.

Solution. Putting $x = -y$ in the given equation, we get

$$(-y)^3 - 3(-y)^2 + 5(-y) - 4 = 0$$
$$\Rightarrow y^3 + 3y^2 + 5y + 4 = 0.$$

Hence, $x^3 + 3x^2 + 5x + 4 = 0$ is the required equation.

Example 4.6. Find the equation whose roots are the roots of the equation $x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$ with their signs changed.

Solution. Putting $x = -y$ in the given equation, we get

$$(-y)^4 - 5(-y)^3 + 7(-y)^2 - 17(-y) + 11 = 0$$
$$\Rightarrow y^4 + 5y^3 + 7y^2 + 17y + 11 = 0.$$

Hence, $x^4 + 5x^3 + 7x^2 + 17x + 11 = 0$ is the required equation.

In-text Exercise 4.3. 1. Find the equation whose roots are the roots of $8x^5 - 5x^4 + 3x^3 + x^2 + x + 8 = 0$ with their sign changed.

2. Find the equation whose roots are the negative of the roots of $x^4 + 21x^3 - x^2 + 10x + 11 = 0$.

4.4.2 To form an equation whose roots are c times the roots of a given equation

Let $p(x) = a_0x + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of $p(x)$ then

$$p(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) \tag{4.7}$$

Put $x = cy$ in (1.1), we have

Therefore the required equation is

$$p\left(\frac{y}{c}\right) = a_0\left(\frac{y}{c} - \alpha_1\right)\left(\frac{y}{c} - \alpha_2\right)\cdots\left(\frac{y}{c} - \alpha_n\right)$$

Thus the roots of $p(\frac{y}{c})$ are $c\alpha_1, c\alpha_2, \cdots, c\alpha_n$. Therefore, the required equation is

$$p(\frac{y}{c}) = a_0\left(\frac{y}{c}\right)^n + a_1\left(\frac{y}{c}\right)^{n-1} + \cdots + a_n = 0$$
$$a_0y^n + ca_1y^{n-1} + c^2a_2y^{n-2} + \cdots + c^na_n = 0.$$

Note. The equation whose roots are c times the roots of a given equation, multiply the coefficients and the constant term by $1, c, c^2, \cdots, c^{n-1}$ and c^n to the given equation arranged in the increasing powers of x.

Example 4.7. Find the equation whose roots are three times those of the equation $x^3 + 2x^2 - 3x + 1 = 0$.

Solution. Let y be the root of the required equation then $y = 3x$. Putting $x = \frac{y}{3}$ in the given equation, we get

$$\left(\frac{y}{3}\right)^3 + 2\left(\frac{y}{3}\right)^2 - \frac{y}{3} + 1 = 0$$
$$\left(\frac{y^3}{27}\right) + 2\left(\frac{y^2}{9}\right) - \frac{y}{3} + 1 = 0.$$
$$y^3 + 6y^2 - 9y + 27 = 0.$$

Hence, $y^3 + 6y^2 - 9y + 27 = 0$ is the required equation.

Example 4.8. Find the equation whose roots are five times those of the equation $4x^4 + 6x^3 + 7x^2 - x + 2 = 0$.

Solution. Let y be the root of the required equation then $y = 5x$. Putting $x = \frac{y}{5}$ in the given equation, we get

$$4\left(\frac{y}{5}\right)^4 + 6\left(\frac{y}{5}\right)^3 + 7\left(\frac{y}{5}\right)^2 - \frac{y}{5} + 2 = 0$$
$$4\left(\frac{y^4}{625}\right) + 6\left(\frac{y^3}{125}\right) + 7\left(\frac{y^2}{25}\right) - \frac{y}{5} + 2 = 0.$$
$$4y^4 + 30y^3 + 175y^2 - 125y + 625 = 0.$$

Hence, $4y^4 + 30y^3 + 175y^2 - 125y + 625 = 0$ is the required equation.

Example 4.9. Remove the fractional coefficient from the equation $x^3 - 4x^2 + \frac{1}{4}x - \frac{1}{9} = 0$.

Solution. Taking $y = cx$ or $x = \frac{y}{c}$ in the given equation, we get

$$\left(\frac{y}{c}\right)^3 - 4\left(\frac{y}{c}\right)^2 + \frac{1}{4}\frac{y}{c} - \frac{1}{9} = 0$$
$$\Rightarrow y^3 - 4y^2c + \frac{1}{4}yc^2 - \frac{1}{9} = 0$$
$$\Rightarrow y^3 - 4y^2c + \frac{1}{2^2}c^2 - \frac{1}{3^2}c^3 = 0.$$

The value of c for which the fraction will disappear is $c = 2 \cdot 3 = 6$. Putting this value of $c = 6$, we get

$$y^3 - 24y^2 + 9y - 24 = 0$$

Hence, the required equation is $x^3 - 24x^2 + 9x - 24 = 0$.

Example 4.10. Transform the equation $72x^3 - 54x^2 + 45x - 7 = 0$ into equation with integral coefficients and unity of the leading coefficient.

Solution. The given equation can be written as

$$x^3 - \frac{54}{72}x^2 + \frac{45}{72}x - \frac{7}{72} = 0.$$

Taking $y = cx$ or $x = \frac{y}{c}$ in the given equation, we get

$$\left(\frac{y}{c}\right)^3 - \frac{54}{72}\left(\frac{y}{c}\right)^2 + \frac{45}{72}\frac{y}{c} - \frac{7}{72} = 0$$
$$\Rightarrow y^3 - \frac{3}{4}y^2c + \frac{5}{8}yc^2 - \frac{7}{72}c^3 = 0.$$

The above equation is with the leading coefficient as unity. If we take $c = 2^2 \cdot 3$, then we get

$$y^3 - 9y^2 + 90y - 168 = 0$$

Hence, the required equation is $x^3 - 9x^2 + 90x - 168 = 0$.

In-text Exercise 4.4.

1. Find the equation whose roots are three times the roots of $x^3 + 11x^2 + 13x + 2 = 0$.
2. Transform the equation $12x^3 - 48x^2 + 56x - 8 = 0$ into equation with integral coefficients and unity of the leading coefficient.

4.4.3 To form an equation whose roots are the reciprocals of the roots of a given equation

Reciprocal Equations are those equations which remains unchanged when x is replaced by its reciprocal.

Let $p(x) = a_0x + a_1x + a_2x^2 + \cdots + a_nx^n = 0$. Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of $p(x)$ then

$$p(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) \tag{4.8}$$

Put $y = \frac{1}{x}$ i.e., $x = \frac{1}{y}$ in (4.8), we have

Therefore the required equation is

$$p\left(\frac{1}{y}\right) = \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\cdots\left(\frac{1}{y} - \alpha_n\right)$$
$$a_0\left(\frac{1}{y}\right)^n + a_1\left(\frac{1}{y}\right)^{n-1} + \cdots + a_n = \frac{a_n}{y_n}\left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\cdots(\frac{1}{y} - \alpha_n)$$
$$\Rightarrow y^n + \frac{a_{n-1}}{a_n}y^{n-1} + \frac{a_{n-2}}{a_n}y^{n-2} + \cdots + \frac{a_1}{a_n}y + \frac{a_0}{a_n} = \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\cdots\left(\frac{1}{y} - \alpha_n\right).$$

Thus the condition obtained are

$$\frac{a_{n-1}}{a_n} = a_1,$$
$$\frac{a_{n-2}}{a_n} = a_2,$$
$$\cdots,$$
$$\frac{1}{a_n} = a_n. \tag{4.9}$$

The last equation (**??**) gives $a_n^2 = 1$. This implies that $a_n = +1$ or $a_n = -1$.

Reciprocal equations are classified into two classes according to $a_n = 1$ or$a_n = -1$.

Case I. First class of reciprocal equations are the equations in which the coefficients of the corresponding terms taken from the beginning and end are equal in magnitude and have the same signs i.e.,

$$a_1 = a_{n-1}, a_2 = a_{n-2}, \cdots, a_{n-1} = a_1.$$

Case II. Second class of reciprocal equations, in which corresponding terms counting from the beginning and end are equal in magnitude but different in signs i.e.,

$$a_1 = -a_{n-1}, a_2 = -a_{n-2}, \cdots, a_{n-1} = -a_1.$$

Note.

1. When the degree of the equation is even then one of the condition becomes $a_m = -a_m$ or $a_m = 0$. So in reciprocal equations of the second class whose degree is even the middle term is absent and roots occur in pairs i.e., $\alpha_1, \frac{1}{\alpha_1}$ etc.

2. When the degree of the equation is odd then there must be a root which is its own reciprocal.

3. Since the reciprocal equation is of first class or second class and -1 or 1 is the root. So the equation can be divided by $x + 1$ or $x - 1$ and outcome is the reciprocal equation of even degree and of the first class.

4. In equations of the second class of even degree $x^2 - 1$ is a factor, since the equation may be written in the form

   $$x^n - 1 + a_1 x(x^{n-2} - 1) + \cdots = 0.$$

   By dividing by $x^2 - 1$, this is also reducible to a reciprocal equation of the first class of even degree. Hence all reciprocal equations may be reduced to those of the first class whose degree is even

Example 4.11. Solve $x^4 - 10x^3 + 16x^2 - 10x + 1 = 0$.

Solution. The given equation is a standard reciprocal equation. Dividing throughout by x^2, we get

$$x^2 - 10x + 16 - \frac{10}{x} + \frac{1}{x^2} = 0$$
$$\Rightarrow (x^2 + \frac{1}{x^2}) - 10(x + \frac{1}{x}) + 16 = 0.$$

Putting $y = x + \frac{1}{x}$, we get

$$\begin{aligned} y^2 - 2 - 10y + 16 &= 0 \\ \Rightarrow y^2 - 10y + 14 &= 0 \\ \Rightarrow (y-5)(y-2) &= 0 \\ \Rightarrow y &= 5 \text{ or } 2. \end{aligned}$$

Case I:

$$\begin{aligned} y &= 5 \\ \Rightarrow x + \frac{1}{x} &= 5 \\ \Rightarrow x^2 - 5x + 1 &= 0 \\ \Rightarrow x &= \frac{5 \pm \sqrt{25-4}}{2} \\ \Rightarrow x &= \frac{5 \pm \sqrt{21}}{2}. \end{aligned}$$

Case II.

$$\begin{aligned} y &= 2 \\ \Rightarrow x + \frac{1}{x} &= 2 \\ \Rightarrow x^2 - 2x + 1 &= 0 \\ \Rightarrow (x-1)^2 &= 0 \\ \Rightarrow x &= 1, 1. \end{aligned}$$

Hence, the roots are $1, 1, \frac{5+\sqrt{21}}{2}, \frac{5-\sqrt{21}}{2}$.

Example 4.12. Solve $5x^3 - 21x^2 - 21x + 5 = 0$.

Solution. This is an odd degree reciprocal equation of first class. Therefore, -1 is the root of the equation and $(x + 1)$ is the factor of the given equation. Dividing the given equation by $(x + 1)$, we obtain $5x^2 - 26x + 5$ as the quotient. Solving the quotient, we obtain the roots as $5, \frac{1}{5}$. Thus $-1, \frac{1}{5}, 5$ are the roots of the given equation.

Example 4.13. Find the equation whose roots are the reciprocal of the roots of the equation $x^4 - 5x^3 + 3x^2 - 2x + 1 = 0$.

Solution. Let y be the root of the required equation then $y = \frac{1}{x}$. Putting $x = \frac{1}{y}$ in the given equation, we get

$$\begin{aligned} &\left(\frac{1}{y}\right)^4 - 5\left(\frac{1}{y}\right)^3 + 3\left(\frac{1}{y}\right)^2 - 2\left(\frac{1}{y}\right) + 1 = 0 \\ &\Rightarrow 1 - 5y + 3y^2 - 2y^3 + y^4 = 0. \end{aligned}$$

Hence, $y^4 - 2y^3 + 3y^2 - 5y + 1 = 0$ is the required equation.

Example 4.14. Find the equation whose roots are the reciprocal of the roots of the equation $5x^5 - 5x^3 - 2x + 4 = 0$.

Solution. Let y be the root of the required equation then $y = \frac{1}{x}$. Putting $x = \frac{1}{y}$ in the given equation, we get

$$\left(5\frac{1}{y}\right)^5 - 5\left(\frac{1}{y}\right)^3 - 2\left(\frac{1}{y}\right) + 4 = 0$$
$$\Rightarrow 4y^5 - 2y^4 - 5y^2 + 5 = 0.$$

Hence, $4x^4 - 2x^3 - 5x + 5 = 0$ is the required equation.

Example 4.15. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 + ax^2 + bx + c = 0$, find the equation whose roots are $\alpha_1^2, \alpha_2^2, \alpha_3^2$.

Solution. Our aim is to find the equation whose roots are square of the roots of

$$x^3 + ax^2 + bx + c = 0.$$

Taking $y = x^2$ in the above equation, we get

$$xy - ay + bx - c = 0$$
$$\Rightarrow (y + b)x = ay + c$$
$$\Rightarrow (y + b)^2x^2 = (ay + c)^2$$
$$\Rightarrow (y + b)^2y = (ay + c)^2$$
$$\Rightarrow (y + b)^2y - (ay + c)^2 = 0.$$

4.4.4 To form an equation whose roots are less by or greater by k then the roots of a given equation

To form an equation whose roots are less by or greater by k then the roots of a given equation. (i.e., Diminishing or increasing the roots by k). Let $p(x) = a_0x + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of $p(x)$ then

$$p(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) \tag{4.10}$$

Put $y = x - k$ i.e., $x = y + k$ in (1.7), we have

Therefore the required equation is

$$p(y + k) = a_0(y + k - \alpha_1)(y + k - \alpha_2)\cdots(y + k - \alpha_n)$$

Thus the roots of $p(y+k)$ are $\alpha_1 - k, \alpha_2 - k, \cdots, \alpha_n - k$. Therefore, the required equation is

$$p(y + k) = a_0(y + k)^n + a_1(y + k)^{n-1} + \cdots + a_n = 0.$$

Expanding using binomial theorem and combining like terms, we get an equation of the form

$$b_0y^n + b_1y^{n-1} + \cdots + b_n = 0. \tag{4.11}$$

Replacing $y = x - k$, we obtain

$$b_0(x-k)^n + b_1(x-k)^{n-1} + \cdots + b_n = 0. \tag{4.12}$$

Now, equation $p(x)$ and (4.12) represents the same equation. Dividing equation (4.12) continuously by $(x - k)$, we obtain the remainders as $b_0, b_1, \cdots, b_n$.

Substituting these in (4.11), we get the required equation.

Example 4.16. Find the equation whose roots are the roots of the equation $4x^3 - 2x^2 + 7x - 3 = 0$ each decreased by 2.

Solution. Let $y = x - 2$ so that $x = y + 2$. Putting the value of x in the given equation, we get

$$\begin{aligned}
&4x^3 - 2x^2 + 7x - 3 = 0\\
&\Rightarrow 4(y+2)^3 - 2(y+2)^2 + 7(y+2) - 3 = 0\\
&\Rightarrow 4y^3 + 24y^2 + 48y + 32 - 2y^2 - 8 - 8y + 7y + 14 - 3 = 0\\
&\Rightarrow 4y^3 + 22y^2 + 47y + 35 = 0.
\end{aligned}$$

Therefore, $4x^3 + 22x^2 + 47x + 35 = 0$ is the required equation.

In-text Exercise 4.5.

1. Find the equation whose roots are the roots of $x^4 + 10x^2 + 3x + 12 = 0$ each decreased by 3.
2. Find the equation whose roots are the roots of $x^3 + x^2 + 4x - 5 = 0$ each decreased by 2.

4.4.5 To form an equation in which some specific term is missing

Let

$$p(x) = a_0x + a_1x + a_2x^2 + \cdots + a_nx^n = 0. \tag{4.13}$$

Suppose it is required to remove the second term of the given equation. Diminish the roots of the equation by k. Put $y = x - k$ or $x = y + k$ in the above equation, we get

$$\begin{aligned}
&a_0(y+k)^n + a_1(y+k)^{n-1} + \cdots + a_n = 0\\
&\Rightarrow a_0y^n + (na_0k + a_1)y^{n-1} + \cdots + a_n = 0.
\end{aligned}$$

Now, to remove the second term from the given equation, we must have $na_0k + a_1 = 0$. This gives $k = -\frac{a_1}{na_0}$.

Example 4.17. Solve the equation $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$ by removing its second term.

Solution. We compute that $h = \frac{a_1}{a_0} = \frac{-12}{4 \cdot 1} = -3$. Replacing x by $(x+3)$ in the given equation, we obtain

$$
\begin{aligned}
&(x+3)^4 - 12(x+3)^3 + 49(x+3)^2 - 78(x+3) + 40 = 0 \\
&\Rightarrow x^4 + 4(3x^3) + 6(9x^2) + 4(27x) + 81 \\
&- 12(x^3 + 3(3x^2) + 3(9x) + 27) + 49(x^2 + 9 + 6x) - 78x - 234 + 40 = 0 \\
&\Rightarrow x^4 + 12x^3 + 54x^2 + 108x + 81 \\
&- 12x^3 - 108x^2 - 324x - 324 + 49x^2 + 441 + 294x - 78x - 234 + 40 = 0 \\
&\Rightarrow x^4 - 5x^2 + (562 - 558) = 0 \\
&\Rightarrow x^4 - 5x^2 + 4 = 0 \\
&\Rightarrow (x^2 - 4)(x^2 - 1) = 0 \\
&\Rightarrow x^2 = 4, 1 \\
&\Rightarrow x = \pm 2, \pm 1.
\end{aligned}
$$

Hence $(x+3)$ gives $1, 5, 4, 2$ as the roots of the given equation.

Example 4.18. Find the equation whose roots are the roots of $x^4 - 5x^3 + 3x^2 - 12x + 11 = 0$ each diminished by 4.

Solution.

$x-2\vert$	1	-5	3	-14	$+15$
		4	-4	12	-20
	1	-1	-1	-2	-5
		4	12	60	
	1	3	11	58	
		4	28		
	1	7	39		
		4			
	1	11			

Hence, the required equation is given by $x^4 + 11x^3 + 39x^2 + 58x - 5 = 0$.

Example 4.19. Find the equation whose roots are the roots of $x^5 - 3x^4 - x^3 + 4x^2 - 13x + 15 = 0$ each diminished by 2.

Solution.

$x-2\|$	1	-3	-1	4	-13	15
		2	-2	-6	-4	-34
	1	-1	-3	-2	-17	-19
		2	2	-2	-8	
	1	1	-1	-4	-25	
		2	6	-10		
	1	3	-5	-14		
		2	10	10		
	1	5	5	-4		
		2	14			
	1	7	19			
		2				
	1	9				

Hence, the required equation is given by $x^5 + 9x^4 + 19x^3 - 4x^2 - 25x + 19 = 0$.

4.5 Summary

1. Cardan's method is used to find the roots of the cubic equation.

2. Descartes' method is used to find the roots of the biquadratic equation.

3. Transformation of equation is transforming one symmetric equation into another in which roots of the new equation has some relation with the roots of the previous equation.

4.6 Self Assessment Exercise

1. Solve the following cubic equations using Cardan's method:

   (a) $x^3 - 2x^2 + 5x + 6 = 0$.

   (b) $2x^3 + x^2 - 8x - 4 = 0$.

   (c) $x^3 - 3x + 12 = 0$.

2. Solve the following biquadratic equations using Descartes' method:

   (a) $x^4 + 4x^3 + 2x^2 + 6x + 8 = 0$.

   (b) $x^4 - 3x^3 + 2x^2 - 11 = 0$.

   (c) $x^4 + 7x^2 + 5x + 3 = 0$.

3. Form an equation whose roots are four times the roots of the equation $x^3 - 6x^2 + 8x - 3 = 0$.

4. Form an equation whose roots are five times the roots of the equation $x^3 + 2x^2 + 3x + 1 = 0$.

5. Form an equation whose roots are the negatives of the roots of the equation $3x^3 - 2x^2 + x - 4 = 0$.

6. Form an equation whose roots are the reciprocal of the roots of the equation $x^4 - 15y^3 - 2y + 11 = 0$.

7. Write an equation whose roots are diminished by 3 of the roots of the equation $x^4 - 5x^3 + 7x^2 - 5x + 1 = 0$.

8. Solve the equation $x^3 + 5x^2 - 4x + 7 = 0$ by removing its second term.

9. Solve the equation $x^4 - 6x^3 + 11x^2 - 6x + 4 = 0$ by removing its second term.

10. Remove the fractional coefficient from the equation

$$x^3 - \frac{2}{3}x^2 + \frac{1}{2}x - 1 = 0.$$

4.7 Solutions to In-text Exercises

Excercise 1.1

1. $3, \frac{1}{2}(-3 + \sqrt{3}i), \frac{1}{2}(-3 - \sqrt{3}i.$

2. $-1, 4, 6.$

Excercise 1.2

1. Roots are $-1 \pm \sqrt{2}, -1 \pm i\sqrt{2}.$

2. Roots are $1, 3, -2 \pm \sqrt{6}.$

Exercise 1.3

1. $-8x^5 + 5x^4 + 3x^3 - x^2 + x - 8 = 0.$

2. $x^4 - 21x^3 - x^2 - 10x + 11 = 0.$

Lesson - 5

Symmetric Functions

Structure

5.1 Learning Objectives

After studying this chapter, student should be able to

- define symmetric functions and can check whether the given function is symmetric or not.
- implement the fundamental theorem on symmetric functions.
- compute symmetric functions.
- better understanding of transformation of equations with the help of symmetric functions.

5.2 Introduction

In algebra, theory of equations is the study of algebraic equations given by a polynomial. Around $18th$ century, there were two questions of major interest first was the roots of the equation and second was the relationship between the roots and the coefficients of the equation. Theory of equations are also studied through the special type of functions called symmetric functions. The theory of Symmetric functions play an important role in the representation theory of Groups and Combinatorics. Many mathematicians uses symmetry function theory to study the permutations and cycle structure. It has many applications in mathematics and mathematical physics like lie algebra, random matrix theory and symmetries.

5.3 Symmetric Functions

Definition 5.1. Symmetric functions of the roots of an equation are those functions in which all the roots are alike involved, so that the expression remains same in the value when any two of the roots are interchanged.

This means that if an expression has two roots say α_1 and α_2 then replacing α_1 by α_2 in the expression or vice-versa, the value of the expression is unchanged or unaltered.

Symmetric functions are denoted by the letter Σ attached to one term of it with the help of which entire expression can be written down. For example, if α_1 and α_2 are the roots of the quadratic equation then $\sum \alpha_1 = \alpha_1 + \alpha_2$ is the symmetric function.

Also, if $\alpha_1, \alpha_2, \alpha_3$ are the roots of the cubic equation then the expression $\Sigma\alpha_1$ represent the symmetric function $\alpha_1+\alpha_2+\alpha_3$ and $\Sigma{\alpha_1}^3{\alpha_2}^3$ means the sum ${\alpha_1}^3{\alpha_2}^3+{\alpha_2}^3{\alpha_3}^3+{\alpha_3}^3{\alpha_1}^3$.

Note. $\sum \alpha_1$, $\sum \alpha_1\alpha_2$, $\alpha_1\alpha_2\alpha_3$ are called the ***elementary symmetric functions.*** In general, the elementary symmetric functions of $\alpha_1, \alpha_2, \cdots, \alpha_n$ are given by $\sum \alpha_1$, $\sum \alpha_1\alpha_2$, $\sum \alpha_1\alpha_2\alpha_3$, $\cdots$, $\alpha_1\alpha_2\cdots\alpha_n$.

Note. $\alpha_1^2\alpha_2 + \alpha_2^2\alpha_3 + \alpha_1^2\alpha_3$ is not a symmetric function as interchanging of α_1 and α_2 will change the given function.

Example 5.1. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - ax^2 + bx - c = 0$, find the values of

1. $\sum \alpha_1^2\alpha_2\alpha_3$
2. $\sum \alpha_1^2\alpha_2^2$
3. $(\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3)(\alpha_3 + \alpha_1)$

Solution.

$$\alpha_1 + \alpha_2 + \alpha_3 = a,$$
$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = b,$$
$$\alpha_1\alpha_2\alpha_3 = c.$$

(i).

$$\sum \alpha_1^2\alpha_2\alpha_3 = \alpha_1^2\alpha_2\alpha_3 + \alpha_2^2\alpha_1\alpha_3 + \alpha_3^2\alpha_1\alpha_2$$
$$\Rightarrow \sum \alpha_1^2\alpha_2\alpha_3 = \alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)$$
$$\Rightarrow \sum \alpha_1^2\alpha_2\alpha_3 = ca.$$

(ii).

$$(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)^2 = \alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2 + 2(\alpha_1^2\alpha_2\alpha_3 + \alpha_2^2\alpha_1\alpha_3 + \alpha_3^2\alpha_1\alpha_2)$$
$$\Rightarrow b^2 = \sum \alpha_1^2\alpha_2^2 + 2\alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)$$
$$\Rightarrow \sum \alpha_1^2\alpha_2^2 + 2 = 2ca - b^2.$$

(ii). $(\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3)(\alpha_3 + \alpha_1) = (a - \alpha_3)(a - \alpha_1)(a - \alpha_2)$.
Since $\alpha_1, \alpha_2, \alpha_3$ are the roots of the given equation, so

$$x^3 - ax^2 + bx - c = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3). \tag{5.1}$$

Putting $x = a$ in (2.1), we get

$$(a - \alpha_1)(a - \alpha_2)(a - \alpha_3) = a^3 - a^3 + ab - c$$
$$\Rightarrow (\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3)(\alpha_3 + \alpha_1) = ab - c.$$

Example 5.2. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 + ax^2 + bx + c = 0$, find the values of

(a) $\sum \frac{1}{\alpha_1}$.

(b) $\sum \alpha_1^2$.

(c) $\sum \alpha_1^2\alpha_2$.

(d) $\sum \alpha_1^3$.

Solution. We have

$$\alpha_1 + \alpha_2 + \alpha_3 = -a,$$
$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = b,$$
$$\alpha_1\alpha_2\alpha_3 = -c.$$

(a). Consider $\sum \frac{1}{\alpha_1} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} = \frac{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1}{\alpha_1\alpha_2\alpha_3} = -\frac{b}{c}$.
(b). Consider $\sum \alpha_1^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$. Now,

$$(\alpha_1 + \alpha_2 + \alpha_3)^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_1\alpha_2 + 2\alpha_2\alpha_3 + 2\alpha_3\alpha_1$$
$$= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)$$
$$\Rightarrow (-a)^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2b$$
$$\Rightarrow \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = a^2 - 2b.$$

(c).

$$\begin{aligned}
&\sum \alpha_1^2\alpha_2 = \alpha_1^2\alpha_2 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1 + \alpha_2^2\alpha_1 + \alpha_1^2\alpha_3 + \alpha_3^2\alpha_1 \\
&\Rightarrow \sum \alpha_1 \sum \alpha_1\alpha_2 = (\alpha_1+\alpha_2+\alpha_3)(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1) \\
&= \sum \alpha_1^2\alpha_2 + 3\alpha_1\alpha_2\alpha_3 \\
&\Rightarrow \sum \alpha_1^2\alpha_2 = \sum \alpha_1 \sum \alpha_1\alpha_2 - 3\alpha_1\alpha_2\alpha_3 \\
&= -ab + 3c.
\end{aligned}$$

(d). Consider

$$\begin{aligned}
&\sum \alpha_1 \sum \alpha_1^2 = (\alpha_1+\alpha_2+\alpha_3)(\alpha_1^2+\alpha_2^2+\alpha_3^2) \\
&= -a(a^2-2b) - (-ab+3c) \\
&= -a^3 - 3ab - 3c \\
&\Rightarrow \alpha_1^3 = \sum \alpha_1 \sum \alpha_1^2 - \sum \alpha_1^2\alpha_2 \\
&= -a(a^2-2b) - (-ab+3c) \\
&= -a^3 - 3ab - 3c.
\end{aligned}$$

Example 5.3. If $\alpha_1, \alpha_2, \alpha_3, \alpha_3$ are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$, find the values of

(a) $\sum \frac{1}{\alpha_1}$.

(b) $\sum \alpha_1^2$.

(c) $\sum \alpha_1^2\alpha_2$.

(d) $\sum \alpha_1^2\alpha_2\alpha_4$.

(e) $\sum \alpha_1^2\alpha_2^2$.

(f) $\sum \alpha_1^4$.

Solution.

$$\begin{aligned}
&\alpha_1+\alpha_2+\alpha_3+\alpha_4 = -a, \\
&\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = b, \\
&\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_4+\alpha_3\alpha_4\alpha_1 = -c \\
&\alpha_1\alpha_2\alpha_3\alpha_4 = d.
\end{aligned}$$

(i).

$$\begin{aligned}
&\sum \frac{1}{\alpha_1} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} \\
&= \frac{\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_4+\alpha_3\alpha_4\alpha_1}{\alpha_1\alpha_2\alpha_3\alpha_4} \\
&= \frac{-c}{d}.
\end{aligned}$$

(ii).

$$\begin{aligned}
&\sum \alpha_1^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 \\
&= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^2 - 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_4\alpha_1 + \alpha_4\alpha_2 + \alpha_1\alpha_4) \\
&= a^2 - bc
\end{aligned}$$

(iii).

$$\begin{aligned}
&\sum \alpha_1 \sum \alpha_2 = \sum \alpha_1^2\alpha_2 + \sum \alpha_1\alpha_2\alpha_3 \\
&\Rightarrow \sum \alpha_1^2\alpha_2 = \sum \alpha_1 \sum \alpha_2 - \sum \alpha_1\alpha_2\alpha_3 \\
&\Rightarrow \sum \alpha_1^2\alpha_2 = -ab + c.
\end{aligned}$$

(iv).

$$\begin{aligned}
&\sum \alpha_1 \sum \alpha_1\alpha_3\alpha_4 = \sum \alpha_1^2\alpha_2\alpha_3 + 4\alpha_1\alpha_2\alpha_3\alpha_4 \\
&\Rightarrow \sum \alpha_1^2\alpha_2\alpha_3 = \sum \alpha_1 \sum \alpha_1\alpha_3\alpha_4 - 4\alpha_1\alpha_2\alpha_3\alpha_4 \\
&= (-a)(-c) - 4d \\
&= ac - 4d.
\end{aligned}$$

(v).

$$\begin{aligned}
&(\sum \alpha_1\alpha_2)^2 = \sum \alpha_1^2\alpha_2^2 + 2\sum \alpha_1^2\alpha_2\alpha_3 + 6\alpha_1\alpha_2\alpha_3\alpha_4 \\
&\Rightarrow \sum \alpha_1^2\alpha_2^2 = (\sum \alpha_1\alpha_2)^2 - 2\sum \alpha_1^2\alpha_2\alpha_3 - 6\alpha_1\alpha_2\alpha_3\alpha_4 \\
&\Rightarrow \sum \alpha_1^2\alpha_2^2 = b^2 - 2(ac - 4d) - 6d \\
&= b^2 - 2ac + 2d.
\end{aligned}$$

(vi).

$$\begin{aligned}
&\sum \alpha_1^2 \sum \alpha_1^2 = \sum \alpha_1^4 + 2\sum \alpha_1^2\alpha_2^2 \\
&\Rightarrow \sum \alpha_1^4 = \sum \alpha_1^2 \sum \alpha_1^2 - 2\sum \alpha_1^2\alpha_2^2 \\
&\Rightarrow \sum \alpha_1^4 = (a^2 - 2b)^2 - 2(b^2 - 2ac + 2d) \\
&= a^4 + 2b^3
\end{aligned}$$

5.4 Fundamental Theorem on Symmetric Functions

Any polynomial symmetric in $x_1, \cdots, x_n$ is equal to an integral rational function, with integral coefficients, of the elementary symmetric functions

$$E_1 = \sum x_1, E_2 = \sum x_1x_2, E_3 = \sum x_1x_2x_3, \cdots, E_n = x_1x_2\cdots x_n$$

and the coefficients of the given polynomial. In particular, any symmetric polynomial with integral coefficients is equal to a polynomial in the elementary symmetric functions with integral coefficients.

The equivalent form of the fundamental theorem is given by

Any polynomial symmetric in the roots of an equation, $x^n - E_1x^{n-1} + E_2x^{n-2} + \cdots + (-1)^n E_n = 0$,

is equal to an integral rational function, with integral coefficients, of the coefficients of the equation and the coefficients of the polynomial.

For $n = 2$,

$E_1(x_1, x_2) = x_1 + x_2, E_2(x_1, x_2) = x_1x_2$.

For $n = 3$,

$x_1^3 + x_2^3 = E(x_1, x_2)^3 - 3E_1(x_1, x_2)E_2(x_1, x_2)$.

Example 5.4. Let us write the symmetric function $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ in terms of elementary symmetric functions.

$$\begin{aligned} E_1 &= \alpha_1 + \alpha_2 + \alpha_3 \\ E_2 &= \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 \\ E_3 &= \alpha_1\alpha_2\alpha_3. \end{aligned}$$

Now,

$$\begin{aligned} &\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (\alpha_1 + \alpha_2 + \alpha_3)^2 - 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) \\ &\Rightarrow \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = E_1^2 - 2E_2. \end{aligned}$$

5.5 Rational Functions Symmetric in all but One of the Roots.

If P is a rational function of the roots of an equation $f(x) = 0$ of degree n and if P is symmetric in $n - 1$ of the roots, then P is equal to a rational function, with integral coefficients, of the remaining root and the coefficients of $f(x)$ and P.

Example 5.5. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 + ax^2 + bx + c = 0$, find the value of $\sum \frac{\alpha_1^2+\alpha_2^2}{\alpha_1+\alpha_2}$.

Solution. $\alpha_1 + \alpha_2 + \alpha_3 = -a, \sum \alpha_1\alpha_2 = b, \alpha_1\alpha_2\alpha_3 = -c.$

$$\sum \frac{\alpha_1^2 + \alpha_2^2}{\alpha_1 + \alpha_2} = \frac{(\alpha_1 + \alpha_2)^2 - 2\alpha_1\alpha_2}{\alpha_1 + \alpha_2} = (\alpha_1 + \alpha_2) - \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2}$$
$$\sum \frac{\alpha_1^2 + \alpha_2^2}{\alpha_1 + \alpha_2} = \sum(\alpha_1 + \alpha_2) - 2\left[\frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2} + \frac{\alpha_2\alpha_3}{\alpha_2 + \alpha_3} + \frac{\alpha_1\alpha_3}{\alpha_1 + \alpha_3}\right]$$
$$= 2\sum \alpha_1 - \frac{2\sum \alpha_1\alpha_2(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3)}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3)}$$
$$= -2a - \frac{2\sum \alpha_1\alpha_2(\sum \alpha_1\alpha_2 + \alpha_1^2)}{(-a - \alpha_1)(-a - \alpha_2)(-a - \alpha_3)}$$
$$= -2a + \frac{2\sum \alpha_1\alpha_2(b + \alpha_1^2)}{(-a - \alpha_1)(-a - \alpha_2)(-a - \alpha_3)}$$
$$= -2a + \frac{2(bb + \alpha_1\alpha_2\alpha_3 \sum \alpha_1)}{(a^3 + a^2 \sum \alpha_1 + a \sum \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3)}$$
$$= -2a + \frac{2(b^2 + ac)}{(a^3 - a^3 + ab - c)}$$
$$= \frac{2b^2 + 4ac - 2a^2b}{(ab - c)}.$$

Example 5.6. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 + ax^2 + bx + c = 0$ then find the equation whose roots are $\alpha_1 - \frac{1}{\alpha_2\alpha_3}, \alpha_2 - \frac{1}{\alpha_3\alpha_1}, \alpha_3 - \frac{1}{\alpha_1\alpha_2}$.

Solution. Let

$$y = \alpha_1 - \frac{1}{\alpha_2\alpha_3} = \alpha_1 - \frac{\alpha_1}{\alpha_1\alpha_2\alpha_3} = \alpha_1(1 + \frac{1}{c})$$
$$\Rightarrow y = x(1 + \frac{1}{c})$$
$$\Rightarrow x = \frac{cy}{1 + c}.$$

Putting this value of x in the given equation, we get

$$\left(\frac{cy}{1 + c}\right)^3 + a\left(\frac{cy}{1 + c}\right)^2 + b\frac{cy}{1 + c} + c = 0$$
$$c^2y^3 + ac(1 + c)y^2 + b(1 + c)^2y + (1 + c)^3 = 0.$$

5.6 Sums of Like Powers of the Roots

If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of the equation $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$
We write $s_1 = \sum \alpha_1, s_2 = \sum \alpha_1^2$ and $s_k = \sum \alpha_1^k$. We obtain

$$s_k + a_1s_{k-1} + a_2s_{k-2} + \cdots + a_ns_{k-n} = 0, k > n. \tag{5.2}$$

Equations given by (2.2) is called ***Newton's Identities***

5.7 Newton's Theorem on the Sums of the powers of the roots

The sums of the similar powers of the roots of an equation can be expressed rationally in terms of the coefficients. In other words,

If $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_1$ are the roots of the equation $x^n + A_1 x_{n-1} + A_2 x_{n-2} + \cdots + A_n = 0$, and $S_k = \alpha_1^k k + \alpha_2^k + \cdots + \alpha_n^k$. Then $S_k + S_{k-1}A_1 + \cdots + A_{k-1}S_1 + kA_k = 0$ if $r \leq n$ and $S_k + S_{k-1}A_1 + \cdots + A_{k-1}S_1 + S_{k-n}A_n = 0$ if $k > n$.

Example 5.7. If α_1 is an imaginary root of the equation $x^7 - 1 = 0$ then form the equation whose roots are $\alpha_1 + \alpha_1^6$, $\alpha_1^2 + \alpha_1^5$, $\alpha_1^3 + \alpha_1^4$.

Solution. Let $a = \alpha_1 + \alpha_1^6$, $b = \alpha_1^2 + \alpha_1^5$, $c = \alpha_1^3 + \alpha_1^4$ be the roots of the new equation. Then the new equation is $(x-a)(x-b)(x-c) = 0$ i.e., $x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc = 0$

Now,

$$
\begin{aligned}
a + b + c &= \alpha_1 + \alpha_1^6 + \alpha_1^2 + \alpha_1^5 + \alpha_1^3 + \alpha_1^4 \\
&= \alpha_1(1 + \alpha_1 + \alpha_1^2 + \alpha_1^3 + \alpha_1^4 + \alpha_1^5) \\
&= \frac{\alpha_1(\alpha_1^6 - 1)}{\alpha_1 - 1} \\
&= \frac{(\alpha_1^7 - \alpha_1)}{\alpha_1 - 1} \\
&= \frac{(1 - \alpha_1)}{\alpha_1 - 1} = -1
\end{aligned}
$$

and

$$
\begin{aligned}
ab + bc + ca &= (\alpha_1 + \alpha_1^6)(\alpha_1^2 + \alpha_1^5) + (\alpha_1^2 + \alpha_1^5)(\alpha_1^3 + \alpha_1^4) + (\alpha_1^3 + \alpha_1^4)(\alpha_1 + \alpha_1^6) \\
&= \alpha_1^3 + \alpha_1^6 + \alpha_1^8 + \alpha_1^{11} + \alpha_1^5 + \alpha_1^6 + \alpha_1^8 + \alpha_1^9 + \alpha_1^4 + \alpha_1^9 + \alpha_1^5 + \alpha_1^{10}.
\end{aligned}
$$

Since α_1 is a root of $x^7 – 1 = 0$, we have $\alpha_1^7 = 1$,

$$
\begin{aligned}
ab + bc + ca &= \alpha_1^3 + \alpha_1^6 + \alpha_1 + \alpha_1^4 + \alpha_1^5 + \alpha_1^6 + \alpha_1 + \alpha_1^2 + \alpha_1^4 + \alpha_1 + \alpha_1^5 + \alpha_1^3 \\
&= 2(\alpha_1 + \alpha_1^2 + \alpha_1^3 + \alpha_1^4 + \alpha_1^5 + \alpha_1^6) \\
&= -2(1) = -2.
\end{aligned}
$$

Next,

$$
\begin{aligned}
abc &= (\alpha_1 + \alpha_1^6)(\alpha_1^2 + \alpha_1^5)(\alpha_1^3 + \alpha_1^4) \\
&= (\alpha_1^3 + \alpha_1^6 + \alpha_1^8 + \alpha_1^1 1)(\alpha_1^3 + \alpha_1^4) \\
&= (\alpha_1^6 + \alpha_1^7 + \alpha_1^9 + \alpha_1^{10} + \alpha_1^{11} + \alpha_1^{12} + \alpha_1^{14} + \alpha_1^{15}) \\
&= \alpha_1^6 + \alpha_1^2 + \alpha_1^3 + \alpha_1^4 + \alpha_1^5 + 1 + \alpha_1) = -1.
\end{aligned}
$$

Thus the required equation is
$x^3 + x^2 – 2x + 1 = 0$.

Example 5.8. Find the value in terms of the coefficients of sum of the reciprocals of the roots of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a^{n-1}x + a_n = 0$.

Solution. We know that

$$\alpha_1\alpha_2\cdots\alpha_{n-1} + \cdots + \alpha_2\alpha_3\cdots\alpha_n = (-1)^{n-1}a_{n-1} \tag{5.3}$$
$$\alpha_1\alpha_2\alpha_3\cdots\alpha_n = (-1)^n a_n. \tag{5.4}$$

Dividing (2.4) by (2.3), we get

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n} = -\frac{a_{n-1}}{a_n}$$
$$\sum \frac{1}{\alpha_1} = -\frac{a_{n-1}}{a_n}.$$

5.8 Theorems relating to Symmetric Functions

1. The sum of the exponents of all the roots in any term of any symmetric function of the roots is equal to the sum of the suffixes in each term of the corresponding value in terms of the coefficients.

The suffix of each coefficient in those equations is equal to the degree in the roots of the corresponding function of the roots ; hence in any product of any powers of the coefficients the sum of the suffixes must be equal to the degree in all the roots of the corresponding function of the roots.

2. When an equation is written with binomial coefficients, the expression in terms of the coefficients for any symmetric function of the roots, which is a function of their differences only, is such that the algebraic sum of the numerical factors of all the terms in it is equal to zero.

Example 5.9. Find the value of the symmetric function $(\alpha_1-\alpha_2)^2+(\alpha_2-\alpha_3)^2+(\alpha_3-\alpha_1)^2$ in terms of the coefficients of the equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$.

Solution. The given equation has the binomial coefficients with the numerical coefficients i.e., $1, 3, 3, 1$.

$$\alpha_1 + \alpha_2 + \alpha_3 = -\frac{3a_1}{a_0}$$
$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \frac{3a_2}{a_0}$$
$$\alpha_1\alpha_3\alpha_2 = \frac{a_3}{a_0}.$$

Consider

$(\alpha_1-\alpha_2)^2+(\alpha_2-\alpha_3)^2+(\alpha_3-\alpha_1)^2=\alpha_1^2+\alpha_2^2-2\alpha_1\alpha_2+\alpha_2^2+\alpha_3^2-2\alpha_2\alpha_3+\alpha_3^2+\alpha_1^2-2\alpha_1\alpha_3$
$=2\alpha_1^2+2\alpha_2^2+2\alpha_3^2-2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)$
$=2(\alpha_1^2+\alpha_2^2+\alpha_3^2)-2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)$
$=2(\alpha_1+\alpha_2+\alpha_3)^2-4(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)-2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)$
$=2(-\frac{3a_1}{a_0})^2-6(3\frac{a_2}{a_0})$
$(\alpha_1-\alpha_2)^2+(\alpha_2-\alpha_3)^2+(\alpha_3-\alpha_1)^2=18\frac{a_1^2}{a_0^2}-18\frac{a_2}{a_0}$
$\Rightarrow a_0^2(\alpha_1-\alpha_2)^2+(\alpha_2-\alpha_3)^2+(\alpha_3-\alpha_1)^2=18(a_1^2-a_0a_2).$

Example 5.10. Express $(2\alpha_1-\alpha_2-\alpha_3)(2\alpha_2-\alpha_1-\alpha_3)(2\alpha_3-\alpha_1-\alpha_2)$ in terms of coefficients of the equation $a_0x^3+3a_1x^2+3a_2x+a_3=0$.

Solution.

$$2\alpha_1-\alpha_2-\alpha_3=3\alpha_1-\alpha_1-\alpha_2-\alpha_3=3\alpha_1+\frac{3a_1}{a_0},$$
$$2\alpha_2-\alpha_1-\alpha_3=3\alpha_2+\frac{3a_1}{a_0},$$
$$2\alpha_3-\alpha_1-\alpha_2=3\alpha_3+\frac{3a_1}{a_0}.$$

$$(2\alpha_1-\alpha_2-\alpha_3)(2\alpha_2-\alpha_1-\alpha_3)(2\alpha_3-\alpha_1-\alpha_2)=(3\alpha_1+\frac{3a_1}{a_0})(3\alpha_2+\frac{3a_1}{a_0})$$
$$+(3\alpha_3+\frac{3a_1}{a_0})$$
$$=(9\alpha_1\alpha_2+9\alpha_1\frac{a_1}{a_0}+9\alpha_2\frac{a_1}{a_0}+9\frac{a_1^2}{a_0^2})(3\alpha_3+3\frac{a_1}{a_0})$$
$$=27\alpha_1\alpha_2\alpha_3+27\alpha_1\alpha_2\frac{a_1}{a_0}+27\alpha_1\alpha_3\frac{a_1^2}{a_0^2}$$
$$+27\alpha_1\frac{a_1^2}{a_0^2}+27\alpha_2\alpha_3\frac{a_1^2}{a_0^2}+27\alpha_2\frac{a_1^2}{a_0^2}+27\alpha_3\frac{a_1^2}{a_0^2}+27\frac{a_1^3}{a_0^3}$$
$$=27\frac{a_3}{a_0}+27\frac{a_1^2}{a_0^2}(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)$$
$$+27\frac{a_1^2}{a_0^2}(\alpha_1+\alpha_2+\alpha_3)+27\frac{a_1^3}{a_0^3}$$
$$=27\frac{a_3}{a_0}+27\frac{a_1^2}{a_0^2}(3\frac{a_2}{a_0})+27\frac{a_1^2}{a_0^2}(-3\frac{a_1}{a_0})++27\frac{a_1^3}{a_0^3}$$
$$=27\frac{a_3}{a_0}+81\frac{a_1^2}{a_0^2}(\frac{a_2}{a_0})-27(2\frac{a_1^3}{a_0^3})$$
$$\Rightarrow a_0^3(2\alpha_1-\alpha_2-\alpha_3)(2\alpha_2-\alpha_1-\alpha_3)(2\alpha_3-\alpha_1-\alpha_2)=27(a_3a_0^2+3a_1^2a_0-2a_1^3).$$

5.9 Computation of Symmetric Functions

Sometimes $\sum$-functions involves large number of roots with small exponents, in such cases simpler symmetric functions are multipliecd together to obtain the $\sum$-functions.

For example, to find $\sum x_1^2 x_2^2 x_3 x_4$, we have
$E_2 E_4 = \sum x_1 x_2 \cdot \sum x_1 x_2 x_3 x_4$.

5.10 Summary

5.11 Self Assessment Excercise

1. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the values of
   (a) $\sum \left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right)$.
   (b) $\sum \alpha_1^2$.
   (c) $\sum \alpha_1^3$.
2. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the values of $\sum \frac{\alpha_1^2 + \alpha_2^2}{\alpha_1 + \alpha_2}$.
3. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 + px + q = 0$, find the values of
   (a) $\sum(\alpha_1 + \alpha_2)^2$
   (b) $\sum \frac{1}{\alpha_1 + \alpha_2}$
   (c) $\sum \alpha_1^2 \alpha_2^2$
   (d) $\sum \left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right)$

Lesson - 6

Transformation by Symmetric Functions

Structure

6.1 Learning Objectives

After studying this chapter, student should be able to

- define symmetric functions and can check whether the given function is symmetric or not.
- implement the fundamental theorem on symmetric functions.
- compute symmetric functions.
- better understanding of transformation of equations with the help of symmetric functions.

6.2 Introduction

In algebra, theory of equations is the study of algebraic equations given by a polynomial. Around $18th$ century, there were two questions of major interest first was the roots of the equation and second was the relationship between the roots and the coefficients of the equation. Theory of equations are also studied through the special type of functions called symmetric functions. The theory of Symmetric functions play an important role in the representation theory of Groups and Combinatorics. Many mathematicians uses symmetry function theory to study the permutations and cycle structure. It has many applications in mathematics and mathematical physics like lie algebra, random matrix theory and symmetries.

6.3 Transformation by Symmetric Functions

Sometimes without knowing the roots of an equation in terms of its coefficients, we can transform one symmetric equation into another in which roots of the new equation has some relation with the roots of the previous equation.

Suppose it is required to transform an equation into another whose roots are rational functions of the roots of the given equation. Let the given function $\phi(\alpha_1, \alpha_2, \alpha_3, \cdots)$ contain all the roots. The transformed equation consist of all the possible combinations $\phi(\alpha_1, \alpha_2, \alpha_3), \phi(\alpha_1, \alpha_2, \alpha_4)$ etc given by

$$(y - \phi(\alpha_1, \alpha_2, \alpha_3)(y - \phi(\alpha_1, \alpha_2, \alpha_4) = 0.$$

Example 6.1. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 + ax^2 + bx + c = 0$, find the equation whose roots are $\alpha_1^2, \alpha_2^2, \alpha_3^2$.

Solution. Our aim is to find the equation whose roots are square of the roots of

$$x^3 + ax^2 + bx + c = 0.$$

Suppose the transformed equation be $y^3 + Ax^2 + Bx + C = 0$. Then we have

$$\begin{aligned} -A &= \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \\ B &= \sum \alpha_1^2\alpha_2^2, \\ -C &= \alpha_1^2\alpha_2^2\alpha_3^2. \end{aligned}$$

Now, we have to form the symmetric functions $\sum \alpha_1^2, \sum \alpha_1^2\alpha_2^2$ of the given equation. We obtain

$$\begin{aligned} \sum \alpha_1^2 &= a^2 - 2b, \\ \sum \alpha_1^2\alpha_2^2 &= b^2 - 2ac, \\ \alpha_1^2\alpha_2^2\alpha_3^2 &= c^2. \end{aligned}$$

Therefore, the transformed equation is
$y^3 - (a^2 - 2b)y^2 + (b^2 - 2ac)y - c^2 = 0..$

Example 6.2. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 + ax^2 + bx + c = 0$, find the equation whose roots are $\alpha_1^3, \alpha_2^3, \alpha_3^3$.

Solution. Our aim is to find the equation whose roots are cube of the roots of

$$x^3 + ax^2 + bx + c = 0.$$

Suppose the transformed equation be $y^3 + Ax^2 + Bx + C = 0$. Then we have

$$\begin{aligned} -A &= \alpha_1^3 + \alpha_2^3 + \alpha_3^3, \\ B &= \sum \alpha_1^3\alpha_2^3, \\ -C &= \alpha_1^3\alpha_2^3\alpha_3^3. \end{aligned}$$

Now, we have to form the symmetric functions $\sum \alpha_1^3, \sum \alpha_1^3\alpha_2^3$ of the given equation. We obtain

$$\begin{aligned} \sum \alpha_1^3 &= a^3 - 3ab + 3c, \\ \sum \alpha_1^3\alpha_2^3 &= b^3 - 3abc + 3c^2, \\ \alpha_1^3\alpha_2^3\alpha_3^3 &= -c^3. \end{aligned}$$

Therefore, the transformed equation is
$y^3 - (a^3 - 3ab + 3c)y^2 + (b^3 - 3abc + 3c^2)y + c^3 = 0..$

Example 6.3. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$, find the equation whose roots are $\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2$.

Solution. Our aim is to find the equation whose roots are square of the roots of

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

Suppose the transformed equation be $y^3 + Ax^3 + Bx^2 + Cx + D = 0$. Then we have

$$\begin{aligned} -A &= \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \\ B &= \sum \alpha_1^2\alpha_2^2, \\ -C &= \sum \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2, \\ D &= \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2. \end{aligned}$$

Now, we have to form the symmetric functions $\sum \alpha_1^2, \sum \alpha_1^2\alpha_2^2, \sum \alpha_1^2\alpha_2^2\alpha_3^2, \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2$ of the given equation. We obtain

$$\begin{aligned} \sum \alpha_1^2 &= a^2 - 2b, \\ \sum \alpha_1^2\alpha_2^2 &= b^2 - 2ac + 2d, \\ \sum \alpha_1^2\alpha_2^2\alpha_3^2 &= c^2 - 2bd, \\ \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2 &= d^2. \end{aligned}$$

Therefore, the transformed equation is
$y^4 - (a^2 - 2b)y^3 + (b^2 - 2ac + 2d)y^2 - (c^2 - 2bd)y + d^2 = 0.$

Example 6.4. Form an equation whose roots are the squares of the roots of the equation $x^3 + x^2 + 7x + 4 = 0$.

Solution. Let $y = x^2$. The given equation may be written as

$$\begin{aligned} y \cdot x + y + 7x + 4 &= 0 \\ \Rightarrow y \cdot x + 7x &= -y - 4 \\ \Rightarrow x(y+7) &= -1(y+4). \end{aligned} \tag{6.1}$$

Squaring both the sides of (3.1), we get

$$\begin{aligned} x^2(y+7)^2 &= (y+4)^2 \\ \Rightarrow y(y+7)^2 &= (y+4)^2 \\ \Rightarrow y(y^2+14y+49) &= y^2+8y+16 \\ \Rightarrow y^3+14y^2+49y &= y^2+8y+16 \\ \Rightarrow y^3+13y^2+41y-16 &= 0. \end{aligned}$$

Hence, $y^3 + 13y^2 + 41y - 16 = 0$ is the required equation.

In-text Exercise 6.1.

1. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - ax^2 + bx + c = 0$ then form an equation whose roots are $\alpha_1^2, \alpha_2^2, \alpha_3^2$.

2. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - ax^2 + bx - c = 0$ then form an equation whose roots are $\alpha_1^3, \alpha_2^3, \alpha_3^3$.

3. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - px^2 + qx^2 - rx + s = 0$ then form an equation whose roots are $\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2$.

4. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - px^2 + qx^2 + rx + s = 0$ then form an equation whose roots are $\alpha_1^3, \alpha_2^3, \alpha_3^3, \alpha_4^3$.

5. Find the equation whose roots are the cubes of the roots of the equation $x^3 + x^2 - 4x + 6 = 0$.

6. Find the equation whose roots are the squares of the roots of the equation $x^4 - 5x - 7 = 0$.

6.4 Transformation in General

In general case, aim to find a new equation where the roots of the new equation are related by a given relation $g(x, y) = 0$ to the roots of the given equation $f(x) = 0$. We will obtain the transformed equation by substituting the value of x in terms of y with the help of $g(x, y) = 0$.

Example 6.5. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 - bx^2 + cx - d = 0$, find the equation whose roots are $\alpha_2\alpha_3 + \frac{1}{\alpha_1}, \alpha_3\alpha_1 + \frac{1}{\alpha_2}, \alpha_1\alpha_2 + \frac{1}{\alpha_3}$.

Solution. Let $y = \alpha_2\alpha_3 + \frac{1}{\alpha_1} = \frac{\alpha_1\alpha_2\alpha_3+1}{\alpha_1} = \frac{d+1}{\alpha_1}$.

Thus we have $y = \frac{d+1}{x}$. This gives that $x = \frac{d+1}{y}$. Substituting this value of x in the given equation, we get

$$\left(\frac{d+1}{y}\right)^3 - b\left(\frac{d+1}{y}\right)^2 + c\left(\frac{d+1}{y}\right) - d = 0$$
$$\Rightarrow (d+1)^3 - b(d+1)y + c(d+1)y^2 - dy^3 = 0$$
$$\Rightarrow dy^3 - c(d+1)y^2 + b(d+1)y - (d+1)^3 = 0.$$

Hence, $dx^3 - c(d+1)x^2 + b(d+1)x - (d+1)^3 = 0$ is the required equation.

Example 6.6. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 + ax^2 + bx + c = 0$, find the equation whose roots are $\alpha_1 - \frac{1}{\alpha_2\alpha_3}, \alpha_2 - \frac{1}{\alpha_3\alpha_1}, \alpha_3 - \frac{1}{\alpha_1\alpha_2}$.

Solution. Let $y = \alpha_1 - \frac{1}{\alpha_2\alpha_3} = \alpha_1 - \alpha_1\frac{1}{\alpha_1\alpha_2\alpha_3} = \alpha_1\left(1 + \frac{1}{c}\right)$.

Thus we have $y = x(1 + \frac{1}{c})$. This gives that $x = \frac{cy}{1+c}$. Substituting this value of x in the given equation, we get

$$\left(\frac{cy}{1+c}\right)^3 + a\left(\frac{cy}{1+c}\right)^2 + b\left(\frac{cy}{1+c}\right) + c = 0$$
$$\Rightarrow (cy)^3 + ac^2(1+c)y^2 + bc(1+c)y^2 - cy^3 = 0$$
$$\Rightarrow c^2y^3 + ac(1+c)y^2 + b(1+c)^2y + (1+c)^3 = 0.$$

Hence, $c^2x^3 + ac(1+c)x^2 + b(1+c)^2x + (1+c)^3 = 0$ is the required equation.

Example 6.7. Find the value of $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ for the given cubic equation

$$x^3 - ax^2 + bx - c = 0.$$

Solution.

$$\alpha_1 + \alpha_2 + \alpha_3 = -a,$$
$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = b$$
$$\Rightarrow (\alpha_1 + \alpha_2 + \alpha_3)^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_1\alpha_2 + 2\alpha_2\alpha_3 + 2\alpha_3\alpha_1$$
$$= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)$$
$$\Rightarrow a^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2b$$
$$\Rightarrow \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = a^2 - 2b.$$

Example 6.8. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, then find the equation whose roots are $\frac{\alpha_1}{\alpha_2+\alpha_3-\alpha_1}, \frac{\alpha_2}{\alpha_3+\alpha_1-\alpha_2}, \frac{\alpha_3}{\alpha_1+\alpha_2-\alpha_3}$.

Solution. $\alpha_1 + \alpha_2 + \alpha_3 = -b, \alpha_1\alpha_2\alpha_3 = -c$. Take

$$y = \frac{\alpha_1}{\alpha_2 + \alpha_3 - \alpha_1} = \frac{\alpha_1}{-a - \alpha_1 - \alpha_1} = \frac{\alpha_1}{-a - 2\alpha_1}$$
$$\Rightarrow x = -\frac{ay}{1+2y}.$$

Putting the value of x in the given equation, we get

$$(-\frac{ay}{1+2y})^3 + a(-\frac{ay}{1+2y})^2 + b(-\frac{ay}{1+2y}) + c = 0$$
$$-a^3y^3 + a32y^2(1+2y) - aby(1+2y)^2 + c(1+2y)^3 = 0.$$

In-text Exercise 6.2.

1. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3+ex^2+fx+g=0$, then find the equation whose roots are $\alpha_1^2+2\alpha_2\alpha_3, \alpha_2^2+2\alpha_3\alpha_1, \alpha_3^2+2\alpha_1\alpha_2$.
2. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3-qx^2+rx-s=0$, then find the equation whose roots are $\alpha_2\alpha_3-\alpha_1^2, \alpha_3\alpha_1-\alpha_2^2, \alpha_1\alpha_2-\alpha_3^2$.
3. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3+ax^2+b=0$, form an equation whose roots are $\alpha_2+\alpha_3, \alpha_3+\alpha_1, \alpha_1+\alpha_2$.
4. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3+kx^2+m=0$, form an equation whose roots are $\frac{\alpha_2\alpha_3}{\alpha_1}, \frac{\alpha_3\alpha_1}{\alpha_2}, \frac{\alpha_1\alpha_2}{\alpha_3}$.

6.5 Equation of Differences in General

The general problem of the formulation of the equation whose roots are the differences or the square of the differences of the roots of the given equation.

Example 6.9. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3-6x^2+11x-6=0$ then find the equation whose roots are $\alpha_1^2+\alpha_2^2, \alpha_2^2+\alpha_3^2, \alpha_3^2+\alpha_1^2$.

Solution. We have $\alpha_1+\alpha_2+\alpha_3=6, \alpha_1\alpha_2\alpha_3=6$.

Let

$$y = \alpha_2^2+\alpha_3^2 \tag{6.2}$$
$$\Rightarrow y = (\alpha_2+\alpha_3)^2 - 2\alpha_2\alpha_3 \tag{6.3}$$
$$\Rightarrow y = (6-\alpha_1)^2 - 12/\alpha_1$$
$$\Rightarrow \alpha_1 y = \alpha_1(6-\alpha_1)^2 - 12$$
$$\Rightarrow \alpha_1^3 - 12\alpha_1^2 + (36-y)\alpha_1 - 12 = 0 \tag{6.4}$$

Since α_1 satisfies the given equation, so

$$\alpha_1^3 - 6\alpha_1^2 + 11\alpha_1 - 6 = 0. \tag{6.5}$$

Subtracting (3.2) from (3.1), we get

$$6\alpha_1^2 + (y-25)\alpha_1 + 36 = 0 \tag{6.6}$$

Example 6.10. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3+bx+c=0$ then find the equation whose roots are $\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1}{\alpha_2}, \frac{\alpha_3}{\alpha_2}+\frac{\alpha_2}{\alpha_3}, \frac{\alpha_3}{\alpha_1}+\frac{\alpha_1}{\alpha_3}$.

Solution. Let $y = \frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$. Then

$$\begin{aligned}&\alpha_1+\alpha_2+\alpha_3=0\\ &\Rightarrow -\alpha_3)=\alpha_1+\alpha_2\\ &\Rightarrow (\alpha_3)^2=(\alpha_1+\alpha_2)^2\\ &\Rightarrow \alpha_3^2=\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2\\ &\Rightarrow \alpha_3^2-2\alpha_1\alpha_2=\alpha_1^2+\alpha_2^2.\end{aligned}$$

This gives that $y = \frac{\alpha_3^2-2\alpha_1\alpha_2}{\alpha_1\alpha_2} = \frac{\alpha_3^2}{\alpha_1\alpha_2} - 2 = \frac{\alpha_3^3}{\alpha_1\alpha_2\alpha_3} - 2 = -\frac{\alpha_3^3}{c} - 2$. This implies that $c(y+2) = -\alpha_3^3$. We take $x^3 = -c(y+2)$.

Putting the value of x^3 in the given equation, we get

$$\begin{aligned}&-c(y+2)+bx+c=0\\ &\Rightarrow bx-cy-c=0\\ &\Rightarrow bx=c(y+1)\\ &\Rightarrow b^3x^3=c^3(y+1)^3\\ &\Rightarrow -b^3c(y+2)=c^3(y+1)^3\\ &\Rightarrow -b^3(y+2)=c^2(y^3+1+3x^2+3x)\\ &\Rightarrow c^2(y^3+3y^2+3y+1)+b^3(y+2)=0\\ &\Rightarrow c^2y^3+3c^2y^2+3cy+c^2+b^3y+2b^3=0\\ &\Rightarrow c^2y^3+3c^2y+(3c+b^3)y+c^2+2b^3=0.\end{aligned}$$

Example 6.11. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 + bx + c = 0$ then find the equation whose roots are $l\alpha_1 + m\alpha_3\alpha_2 + l\alpha_2 + m\alpha_3\alpha_1, l\alpha_3 + m\alpha_1\alpha_2$.

Solution. Let $\alpha_1+\alpha_2+\alpha_3 = 0, \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = b, \alpha_1\alpha_2\alpha_3 = -c.$ $y = \frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$.

Let

$$\begin{aligned}&y=l\alpha_1+m\alpha_2\alpha_3\\ &\Rightarrow \alpha_1 y=l\alpha_1^2+\alpha_1\alpha_2\alpha_3\\ &\Rightarrow \alpha_1^2-\alpha_1 y-mc=0\end{aligned}\tag{6.7}$$

Since α_1 is the root of the given equation, therefore

$$\alpha_1^3+b\alpha_1+c=0.\tag{6.8}$$

Multiplying (4) by α_1, (5) by l and subtract, we get

$$y\alpha_1^2+(mc+lb)\alpha_1+lc=0.\tag{6.9}$$

Solving 4,6 for α_1, α_1^2, we have

$$\begin{aligned}&\frac{\alpha_1^2}{-lcy+mc(mc+lb)}=\frac{\alpha_1}{-mcy-l^2c}=\frac{1}{l(mc+lq)+y^2}\\ &\Rightarrow \alpha_1^2=\frac{mc(mc+lb)-lcy}{l(mc+lq)+y^2}, \alpha_1=\frac{-(mcy+l^2c)}{l(mc+lq)+y^2}.\end{aligned}$$

Eliminating α_1, we get

$$\frac{c[m(mc+lb)-ly]}{l(mc+lq)+y^2} = \frac{c^2[(my+l^2)]^2}{[l(mc+lq)+y^2]^2}$$

Hence, $[l(mc+lq)+y^2][m(mc+lb)-ly] = c[(my+l^2)]^2$.

6.6 Summary

6.7 Self Assessment Exercise

1. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - 3x^2 + 5x - 11 = 0$ then find the equation whose roots are $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1$.

2. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 - 7x^2 + 12x - 6 = 0$, form an equation whose roots are $\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2}, \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}, \frac{1}{\alpha_3^2} + \frac{1}{\alpha_1^2}$.

3. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $x^3 + 3x^2 + 2x - 4 = 0$ then find the equation whose roots are $\alpha_1^2 + \alpha_2^2 - \alpha_3^2, \alpha_2^2 + \alpha_3^2 - \alpha_1^2, \alpha_3^2 + \alpha_1^2 - \alpha_2^2$.

Suggested Reading

1. Burnside, W.S., Panton, A.W.,The Theory of Equations (11th ed.). Vol. 1. Dover Publications, Inc., 1979.

2. Dickson, Leonard Eugene, First Course in the Theory of Equations. John Wiley and Sons, Inc., 2009